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Characterization of Probability Measures on Hilbert Spaces via Q-Independence

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Abstract

We obtain a characterization for probability measures on a separable Hilbert space X based on linear forms of Q-independent random elements taking values in X. As a special case, we obtain a characterization of probability distributions on R^k through linear functions of Q-independent k-dimensional random vectors.

Key Words : *Q-independence; Probability measure; Characterization; Random element in Hilbert space; Multivariate random vector.*

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1 Introduction

If X_1 and X_2 are independent standard normal random variables, it is known that the ratio X_1/X_2 has the standard Cauchy distribution. However the converse is not true. For instance, let $Y_1 = \frac{1}{X_1}$ and $Y_2 = \frac{1}{X_2}$, then Y_1 and Y_2 are independent random variables but they do not have the standard normal distribution and yet $Y_2/Y_1 = X_1/X_2$ has the standard Cauchy distribution. Kotlarski (1967) has proved that, if X_1, X_2 and X_3 are independent identically distributed random variables such that (Z_1, Z_2) has the bivariate Cauchy distribution where $Z_1 = \frac{X_1}{X_2}$ and $Z_2 = \frac{X_1}{X_3}$,, then the

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random variables X_1, X_2 and X_3 have normal distributions. He proved that if X_1, X_2 and X_3 are three independent real-valued random variables and if the characteristic function of the bivariate random vector (Z_1, Z_2) where $Z_1 = X_1 - X_2, Z_2 = X_1 - X_3$ does not vanish, then the distribution of the random vector (Z_1, Z_2) determines the distributions of the random variables X_1, X_2 and X_3 up to changes in location. Kotlarski's result has found applications in identification and estimation of auction models in economics (cf. Krasnokutskaya (2011)). It can be used when one observes two error-contaminated measurements of the same variable (when the errors are independent). The joint distribution of the contaminated random variables identifies the distributions of the true variable as well as that of the errors up to location. Kotlarski (1966) extended his result to random elements taking values in a Hilbert space. Prakasa Rao (1968) (cf. Prakasa Rao (1992)) generalized the result to random elements taking values in a locally compact Abelian group. Motivation for study of probability measures on Hilbert spaces arises from the intrinsic mathematical interest but also from the applications to functional data analysis where the observations are curves over a specified region, for instance, the observations are functions in the space of square integrable functions $L_2(R)$ which is a Hilbert space. It is now known that functional data analysis has applications in the study of stochastic modeling of trade through e-commerce. Another motivation for study is in the area of signal processing. Suppose $X_3 = \{X_3(t), 0 \leq t \leq T\}, i = 1, 2$ is a signal sent over two different channels and $X_i = \{X_i(t), 0 \leq t \leq T\}, i = 1, 2$ are independent additive components contaminating the original signal X_3 transmitted over these channels. It is of interest to know whether the true signal $X_3 = \{X_3(t), 0 \leq t \leq T\}$ can be recovered from the observed data $\{Z_i(t), 0 \leq t \leq T\}, i = 1, 2$ where $\{Z_1(t) = X_1(t) + X_3(t), 0 \leq t \leq T\}$ and $\{Z_2(t) = X_2(t) + X_3(t), 0 \leq t \leq T\}$. If the processes $X_i, i = 1, 2, 3$ are assumed to have sample paths in the space $L_2[0, T]$, then the processes $X_i, i = 1, 2, 3$ can be considered as random elements taking values in the

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Hilbert space and it follows that the joint probability measure of (Z_1, Z_2) determines the probability measures of $X_i, i = 1, 2, 3$ up to changes in location under some conditions. In a recent article, Kagan and Szekely (2016) introduced the notion of Q-independence for real-valued random variables and studied characterization properties of a Gaussian distribution based on linear forms of Q-independent random variables. It is obvious that independence of random variables implies their Q-independence. However it is known that Q-independence of a set of real-valued random variables does not imply the independence of the set. For instance, if X, Y, Z are non-degenerate independent random variables, then $X + Y$ and $X + Z$ are Q-independent but not independent. Prakasa Rao (2016, 2017) extended Kotlarski's theorem for Q-independent random variables and Q-conditional independent random variables. Our aim in this paper is to extend the result obtained by Kotlarski (1966) to the Q-independent case for Hilbert space valued random elements. As a special case, we obtain a characterization of probability distributions for multi-dimensional random vectors through linear functions of Q-independent random vectors. These results generalize Kotlarski's results from the independent case to the Q-independent case which is a *strictly* larger class of random variables.

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2 Preliminaries

Suppose X is a separable Hilbert space. Let $(x, y), x \in X, y \in X$ denote the inner product between x and y. Let $||x||$ denote the norm of the element $x \in X$. Suppose ψ is a random element defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in the space X and let μ_ψ be the probability measure generated by the random element ψ. The function

$$\hat{\mu}_\psi(y) = \int_X e^{i(x,y)} \mu_\psi(dx), y \in X$$

is the characteristic function of the probability measure μ_ψ. It is known that

(i)the function $\hat{\mu}_\psi(y)$ is a uniformly continuous function of y in the norm topology;
(ii)the function $\hat{\mu}_\psi(.)$ determines the probability measure μ_ψ uniquely;
(iii)$(\mu_\psi \hat{*} \mu_\eta)(y) = \hat{\mu}_\psi(y)\hat{\mu}_\eta(y), y \in X$ where * denotes the convolution operation;
(iv) $\hat{\mu}_\psi(0) = 1$ where 0 is the identity element in X, and
(v) $|\hat{\mu}_\psi(y)| \leq 1, y \in X$.

Properties of probability measures on Hilbert spaces are investigated in Grenander (1963) and Parthasarathy (1967).

Let $f(y)$ be a function defined on the space X and let $h \in X$. Let Δ_h be the finite difference operator defined by

$$\Delta_h f(y) = f(y+h) - f(y).$$

The function $f(y), y \in X$ is called a *polynomial* on X if

$$\Delta_h^{n+1} f(y) = 0$$

for some $n \geq 0$ and for all $y, h \in X$. The minimal n for which this equality holds is called the *degree* of the polynomial $f(y)$. Let $\psi_1, \ldots, \psi_n$ be random elements with values in the Hilbert Space X. Following Kagan and Szekely (2016), we define the notion of Q-independence for random elements $\psi_1, \ldots, \psi_n$ with values in the space X. The random elements $\psi_1, \ldots, \psi_n$, taking values in the Hilbert space X, are said to be *Q-independent* if their joint characteristic function can be represented in the form

$$\hat{\mu}_{(\psi_1,\ldots,\psi_n)}(y_1, \ldots, y_n) = (\Pi_{j=1}^n \hat{\mu}_{\psi_j}(y_j)) \exp[q(y_1, \ldots, y_n)], y_i \in X, 1 \leq i \leq n \tag{2.1}$$

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where $q(y_1, \ldots, y_n)$ is a continuous polynomial on the space X^n with $q(0, \ldots, 0) = 0$. Here X^n denotes the n-fold tensor product of the Hilbert Space X.

3 Main result

We now extend the result proved in Kotlarski (1966) to Q-independent random elements taking values in a Hilbert space X. We will now prove a lemma which will be used in the sequel.

Lemma 3.1: *Let X be a Hilbert space and $b_i, 1 \leq i \leq n$ be scalars such that $b_i \neq b_j \neq 0$ for $i \neq j$. Consider the functional equation*

$$\sum_{j=1}^{n} \psi_j(u + b_j v) = P(u) + Q(v) + R(u, v), u, v \in X \tag{3.1}$$

on the space X where $\psi_j(u), P(u)$ and $Q(u)$ are functions on the Hilbert Space X and $R(u, v)$ is a polynomial on $X \otimes X$. Then $P(y)$ and $Q(y)$ are polynomials on X.

Proof : We use the finite-difference method for proving this lemma (following the techniques in Kagan, Linnik and Rao (1973) and Feldman (2017)). Let h_1 be an arbitrary element in the Hilbert space X. Let $k_1 = -b_n^{-1} h_1$. Then $h_1 + b_n k_1 = 0$. Let us substitute $u + h_1$ for u and $v + k_1$ for v in the equation (3.1). Subtracting the equation (3.1) from the resulting equation, we get that

$$\sum_{j=1}^{n-1} \Delta_{\ell_{1j}} \psi_j(u + b_j v) = \Delta_{h_1} P(u) + \Delta_{k_1} Q(v) + \Delta_{(h_1, k_1)} R(u, v), u, v \in X \tag{3.2}$$

where $\ell_{1j} = h_1 + b_j k_1 = (b_j - b_n) k_1, 1 \leq j \leq (n-1)$. Let h_2 be an arbitrary element of the Hilbert space X. Let $k_2 = -b_{n-1}^{-1} h_2$. Then $h_2 + b_{n-1} k_2 = 0$.

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Substitute $u + h_2$ for u and $v + k_2$ for v in the equation (3.2). Subtracting equation (3.2) from the resulting equation, we obtain that

$$\sum_{j=1}^{n-2} \Delta_{\ell_{2j}} \Delta_{\ell_{1j}} \psi_j(u + b_j v) = \Delta_{h_2}\Delta_{h_1}P(u) + \Delta_{k_2}\Delta_{k_1}Q(v) + \Delta_{(h_2,k_2)}\Delta_{(h_1,k_1)}R(u,v) \quad (3.3)$$

for $(u, v) \in X$where $\ell_{2j} = h_2 + b_j k_2 = (b_j - b_{n-1})k_2, 1 \leq j \leq (n-2)$. Proceeding by similar arguments, we get the equation

$$\begin{aligned} &\Delta_{\ell_{n-1,1}}\Delta_{\ell_{n-2,1}} \ldots \Delta_{\ell_{11}}\psi_1(u + b_1 v) \\ &= \Delta_{h_{n-1}}\Delta_{h_{n-2}} \ldots \Delta_{h_1}P(u) \\ &\quad + \Delta_{k_{n-1}}\Delta_{k_{n-2}} \ldots \Delta_{k_1}Q(v) \\ &\quad + \Delta_{(h_{n-1},k_{n-1})}\Delta_{(h_{n-2},k_{n-2})} \ldots \Delta_{(h_1,k_1)}R(u,v) \quad (3.4) \end{aligned}$$

for $u, v \in X$ where h_m are arbitrary elements of X , $k_m = -b_{n-m+1}^{-1}h_m, 1 \leq m \leq n-1$ and $\ell_{mj} = h_m + b_j k_m = (b_j - b_{n-m+1})k_m, 1 \leq j \leq n-m$. Let h_n be an arbitrary element in X. Let $k_n = -b_1^{-1}h_n$. Then $h_n + b_1 k_n = 0$. Substituting $u+h_n$ for u and $v+k_n$ for v in the equation (3.4) and subtracting the equation (3.4) from the resulting equation, we obtain that

$$\begin{aligned} &\Delta_{h_n}\Delta_{h_{n-1}} \ldots \Delta_{h_1}P(u) \\ &\quad + \Delta_{k_n}\Delta_{k_{n-1}} \ldots \Delta_{k_1}Q(v) \\ &\quad + \Delta_{(h_n,k_n)}\Delta_{(h_{n-2},k_{n-2})} \ldots \Delta_{(h_1,k_1)}R(u,v) = 0 \quad (3.5) \end{aligned}$$

for $u, v \in X$. Let h_{n+1} be an arbitrary element of the space X. Substituting h_{n+1} for u in the equation (3.5) and subtracting the equation (3.5) from the

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resulting equation, we observe that

$$\Delta_{h_{n+1}}\Delta_{h_n}\Delta_{h_{n-1}}\dots\Delta_{h_1}P(u)$$
$$+\Delta_{(h_{n+1},0)}\Delta_{(h_n,k_n)}\Delta_{(h_{n-1},k_{n-1})}\dots\Delta_{(h_1,k_1)}R(u,v)=0 \quad (3.6)$$

for $u, v \in X$.If h and k are arbitrary elements of the Hilbert space X, then, for some ℓ,

$$\Delta^{\ell+1}_{(h,k)}R(u,v)=0, u,v\in X \quad (3.7)$$

since $R(u,v)$ is a polynomial on $X \otimes X$. Since $h_m, 1 \leq m \leq n+1$ are arbitrary elements in the space X, we can substitute $h_1 = \dots = h_{n+1} = h$ in the equation (3.6) and apply the operator $\Delta^{\ell+1}_{(h,k)}$ to both the sides of the resulting equation. Equation (3.7) implies that

$$\Delta^{\ell+n+2}_{h}P(u)=0, u,h\in X. \quad (3.8)$$

Hence $P(u)$ is a polynomial on the Hilbert space X. Similar arguments prove that $Q(v)$ is also a polynomial on X.

Remarks : The proof given above follows arguments similar to those given in Kagan et al. (1973) for functions defined on the real line and by Feldman (2017) for functions defined on groups. We have shown that the arguments continue to hold for functions defined on a Hilbert Space and give details here for completeness.

Theorem 3.2: *Let ψ_1, ψ_2 and ψ_3 be three Q-independent random elements taking values in a separable Hilbert space X. Let $Z_1 = \psi_1 + \psi_2$ and $Z_2 = \psi_2 + \psi_3$. If the characteristic function of the random vector (Z_1, Z_2) does not vanish, then the probability measure of the random vector (Z_1, Z_2) determines the characteristic functions of ψ_1, ψ_2, ψ_3 up to multiplication by*

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the exponentials of polynomials.

Proof: Let $\lambda_{(Z_1,Z_2)}$ denote the joint probability measure of the random vector (Z_1, Z_2). Let μ_{ψ_j} denote the probability measure of the random element ψ_j for $j = 1, 2, 3$. The joint characteristic function of the random vector (Z_1, Z_2) is given by

$$\begin{aligned}
\hat{\lambda}_{(Z_1,Z_2)}(u, v) &= E[\exp(i(Z_1, u) + i(Z_2, v))], u, v \in X \\
&= E[\exp(i(\psi_1 + \psi_2, u) + i(\psi_2 + \psi_3, v))], u, v \in X \\
&= E[\exp(i(\psi_1, u) + i(\psi_2, u + v) + i(\psi_3, v))], u, v \in X \\
&= \hat{\mu}_{\psi_1}(u)\hat{\mu}_{\psi_2}(u + v)\hat{\mu}_{\psi_3}(v) \exp[q_1(u, u + v, v)], u, v \in X
\end{aligned}$$

where $q_1(y_1, y_2, y_3)$ is a continuous polynomial on the space $X \otimes X \otimes X$ by the Q-independence of the random elements ψ_1, ψ_2, ψ_3. Suppose that $\eta_i, i = 1, 2, 3$ is another set of Q-independent random elements such that the joint probability measure of the random vector (T_1, T_2) is the same as the joint probability measure of the random vector (Z_1, Z_2) where $T_1 = \eta_1 + \eta_2$ and $T_2 = \eta_2 + \eta_3$. By the calculations described above, it is easy to check that

$$\hat{\lambda}_{(Z_1,Z_2)}(u, v) = \hat{\mu}_{\eta_1}(u)\hat{\mu}_{\eta_2}(u + v)\hat{\mu}_{\eta_3}(v) \exp[q_2(u, u + v, v)], u, v \in X, \quad (3.9)$$

where $q_2(y_1, y_2, y_3)$ is a continuous polynomial on the space $X \otimes X \otimes X$ by the Q-independence of the random elements η_1, η_2, η_3. Since the joint probability measures of the random vectors (Z_1, Z_2) and (T_1, T_2) are the same with non-vanishing characteristic functions, by hypothesis, it follows that $\hat{\mu}_{\psi_j}(u) \neq 0, u \in Y, j = 1, 2, 3$ and $\hat{\mu}_{\eta_j}(v) \neq 0, v \in X, j = 1, 2, 3$ and

$$\begin{aligned}
&\hat{\mu}_{\psi_1}(u)\hat{\mu}_{\psi_2}(u + v)\hat{\mu}_{\psi_3}(v) \exp[q_1(u, u + v, v)] \\
&\quad = \hat{\mu}_{\eta_1}(u)\hat{\mu}_{\eta_2}(u + v)\hat{\mu}_{\eta_3}(v) \exp[q_2(u, u + v, v)] \qquad (3.10)
\end{aligned}$$

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for $u, v \in X$. Let

$$\zeta_i(u) = \log[\frac{\hat{\mu}_{\psi_i}(u)}{\hat{\mu}_{\eta_i}(u)}], u \in X, i = 1, 2, 3 \tag{3.11}$$

where $\log \hat{\mu}_{\psi_i}(u)$ denotes the continuous branch of the logarithm of the characteristic function $\hat{\mu}_{\psi_i}(u)$ with $\log \hat{\mu}_{\psi_i}(0) = 0$. The equations derived above imply that

$$\zeta_1(u) + \zeta_2(u+v) + \zeta_3(v) = q_3(u, u+v, v), u, v \in X \tag{3.12}$$

where $q_3(y_1, y_2, y_3)$ is a continuous polynomial on the space $X \otimes X \otimes X$. Hence

$$\zeta_2(u+v) = -\zeta_1(u) - \zeta_3(v) + q_3(u, u+v, v), u, v \in X.$$

Applying Lemma 3.1, it follows that $\zeta_1(u)$ and $\zeta_3(u)$ are polynomials in $u \in X$. It can be checked that $\zeta_2(u), i = 1, 3$ is also a polynomial in $u \in X$ from the equation (3.11). Hence

$$\hat{\mu}_{\psi_i}(u) = \hat{\mu}_{\eta_i}(u) \exp[q_i(u)], u \in X, i = 1, 2, 3 \tag{3.13}$$

where $q_i(u), i = 1, 2, 3$ are continuous polynomials on X with $q_i(0) = 0, i = 1, 2, 3$. This completes the proof of Theorem 3.2.

Remarks: Results obtained here can be extended to linear forms of n Q-independent random elements as discussed in Prakasa Rao (2017) for linear forms of Q-independent real valued random variables.

4 Special Case

As a special case of the results obtained in the previous section, we now obtain characterizations for probability measures for Q-independent multi-dimensional random vectors.

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Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be k-dimensional random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\phi_i(\mathbf{t})$ denote the joint characteristic of the random vector $\mathbf{X_i}, i = 1, \ldots, n$. The collection $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is said to be *Q-independent* if the joint characteristic function of the random vector $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ can be represented as

$$\phi_{(\mathbf{X}_1,\ldots,\mathbf{X}_n)}(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \Pi_{i=1}^n \phi_i(\mathbf{t}_i) \exp[q(\mathbf{t}_1, \ldots, \mathbf{t}_n)], \mathbf{t}_1, \ldots, \mathbf{t}_n \in R^k$$

where $q(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ is a polynomial in the components of $\mathbf{t}_1, \ldots, \mathbf{t}_n$. Two random vectors $\mathbf{X}_j$ and $\mathbf{X}_k$ are said to be *Q-identically distributed* if

$$\phi_j(\mathbf{t}) = \phi_k(\mathbf{t}) \exp[q(\mathbf{t})]$$

where $q(\mathbf{t})$ is a polynomial in the components of the vector $\mathbf{t}$. It is known that two random variables could be Q-independent but not independent. For instance, if X, Y, Z are non-degenerate independent Gaussian random variables, then $X + Y$ and $X + Z$ are Q-independent but not independent.

As a consequence of Theorem 3.2, we get the following result characterizing probability measures on the space R^k.

Theorem 4.1: *Let $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ be three Q-independent k-dimensional random vectors. Let $\mathbf{Z}_1 = \mathbf{X}_1 + \mathbf{X}_2$ and $\mathbf{Z}_2 = \mathbf{X}_2 + \mathbf{X}_3$. If the characteristic function $\phi_{(\mathbf{Z}_1,\mathbf{Z}_2)}(\mathbf{t}_1, \mathbf{t}_2)$ of the $2k$-dimensional random vector $(\mathbf{Z}_1, \mathbf{Z}_2)$ does not vanish, then the characteristic function of the random vector $(\mathbf{Z}_1, \mathbf{Z}_2)$ determines the characteristic functions of the k-dimensional random vectors $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ up to multiplication by the exponentials of polynomials in the components of $\mathbf{t}_1, \mathbf{t}_2$.*

Remarks : This result can be extended to n k-dimensional Q-independent random vectors generalizing Theorem 3.3 in Prakasa Rao (2017).

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