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Instrumental Variable Estimation for a Linear Stochastic Differential Equation Driven by a Mixed Fractional Brownian Motion

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Abstract: We investigate the asymptotic properties of instrumental variable estimators of the drift parameter for stochastic processes satisfying linear stochastic differential equations driven by mixed fractional Brownian motion.

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1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

(1. 1) $$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. We discussed more general classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion and studied the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes in Prakasa Rao (2003, 2005). Prakasa Rao (2010) gives a comprehensive discussion on problems of estimation for processes driven by a fractional Brownian motion. So (2005) noted that the standard asymptotic theory developed for studying the asymptotic properties of maximum likelihood estimators (MLE) for diffusion processes (cf. Basawa and Prakasa Rao (1980); Prakasa Rao (1999a,b)) depends on the stationarity of the underlying process $\{X_t, t \geq 0\}$ and hence proposed an instrumental variable approach for the estimation of drift parameter for stochastic processes satisfying linear stochastic differential equations driven by a Wiener process. As discussed by So (2005), this approach allows one to study asymptotic properties of estimators in non-stationary models such as random walk and cointegration which are used in economics and finance literature. We extended this approach in Prakasa Rao (2007) to estimate the parameters for stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion in view of their applications for modeling in finance.

Geometric Brownian motion has been widely used for modeling fluctuations of share prices in a stock market. Recently there has been an interest to study the problem of estimation of parameters for processes driven by processes which are mixtures of independent Brownian and fractional Brownian motions starting from the work of Cheridito (2001), Rudomino-Dusyatska (2003) and more recently in Prakasa Rao (2015a,b) among others. Mixed fractional Brownian models were studied in Mishura (2008) and Prakasa Rao (2010). Cai et al. (2016) present a new approach via filtering for analysis of mixed processes of type $\{X_t = B_t + G_t, 0 \leq t \leq T\}$ where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion and $\{G_t, 0 \leq t \leq T\}$ is an independent Gaussian process. Statistical Analysis of mixed fractional Ornstein-Uhlenbebeck process was investigated in Chigansky and Kleptsyna (2015). Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process were studied by Marushkevych (2016).

Our aim in this paper is to extend the instrumental variable approach for estimation of parameters involved in processes driven by mixed fractional Brownian motion (mFBm) generalizing the work in Prakasa Rao (2007) and So (2005).

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$(2.\ 1) \qquad E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0.$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the mixed fractional Brownian motion with Hurst index H. We assume here after that Hurst index H is known. Following the results in Cheridito (2001), it is known that the process $\tilde{W}^H$ is a semimartingale in its own filtration if and only if either $H = 1/2$ or $H \in (\frac{3}{4}, 1]$. We will assume here after that $H \in (\frac{3}{4}, 1]$.

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$(2.\ 2) \qquad Y_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0$$

where the process $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$(2.\ 3) \qquad dY_t = C(t)dt + d\tilde{W}_t^H, t \geq 0$$

driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai et al.(2016) and Chigansky and Kleptsyna (2016), one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale M^H. Let $g_H(s,t)$ be the solution of the integro-differential equation

$$(2.\ 4) \qquad g_H(s,t) + H\frac{d}{ds}\int_0^t g_H(r,t)|s-r|^{2H-1} sign(s-r)dr = 1, 0 < s < t.$$

Cai et al. (2016) proved that the process

$$(2.\ 5) \qquad M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0$$

is a Gaussian martingale with quadratic variation

$$(2.\ 6) \qquad < M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0$$

Furthermore the natural filtration of the martingale M^H coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. Suppose that, for the martingale M^H defined by the equation (2.5), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

$$Q_t = \frac{d}{d < M^H >_t} \int_0^t g_H(s,t) C(s) ds, t \geq 0 \tag{2. 7}$$

is well defined. Define the process

$$Z_t = \int_0^t g_H(s,t) dY_s, t \geq 0. \tag{2. 8}$$

As a consequence of the results in Cai et al. (2016), it follows that the process Z is a fundamental semimartingale associated with the process Y in the following sense.

Theorem 2.1: Let $g_H(s,t)$ be the solution of the equation (2.4). Define the process Z as given in the equation (2.8). Then the following relations hold.
(i) The process Z is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_s d < M^H >_s + M_t^H, t \geq 0 \tag{2. 9}$$

where M^H is the martingale defined by the equation (2.5).
(ii) The process Y admits the representation

$$Y_t = \int_0^t \hat{g}_H(s,t) dZ_s, t \geq 0 \tag{2. 10}$$

where

$$\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H >_s} \int_0^t g_H(r,s) dr. \tag{2. 11}$$

(iii) The natural filtrations $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ of the processes Y and Z respectively coincide.

Applying Corollary 2.9 in Cai et al. (2016), it follows that the probability measures μ_Y and $\mu_{\tilde{W}^H}$ generated by the processes Y and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

$$\frac{d\mu_Y}{d\mu_{\tilde{W}^H}}(Y) = \exp[\int_0^T Q_s dZ_s - \frac{1}{2} \int_0^T Q_s^2 d < M^H >_s] \tag{2. 12}$$

which is also the likelihood function based on the observation $\{Y_s, 0 \leq s \leq T.\}$ Since the filtrations generated by the processes Y and Z are the same, the information contained in the families of σ-algebras $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observation $\{Y_s, 0 \leq s \leq T\}$ and $\{Z_s, 0 \leq s \leq T\}$ are equivalent.

3 Instrumental Variable Estimation

Let us consider the stochastic differential equation

(3. 1) $$dX(t) = [a(t, X(t)) + \theta\, b(t, X(t))]dt + d\tilde{W}_t^H, X(0) = 0, t \geq 0$$

where $\theta \in \Theta \subset R$ with known Hurst parameter H. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

(3. 2) $$X(t) = \int_0^t [a(s, X(s)) + \theta\, b(s, X(s))]ds + \tilde{W}_t^H, t \geq 0.$$

Let

(3. 3) $$C(\theta, t) = a(t, X(t)) + \theta\, b(t, X(t)), t \geq 0$$

and assume that the sample paths of the process $\{C(\theta, t), t \geq 0\}$ are smooth enough so that the process

(3. 4) $$Q_{H,\theta}(t) = \frac{d}{d < M^H >_t} \int_0^t g_H(s, t) C(\theta, s) ds, t \geq 0$$

is well defined. Suppose the sample paths of the process $\{Q_{H,\theta}, 0 \leq t \leq T\}$ belong almost surely to $L^2([0, T], dw_t^H)$. Define

(3. 5) $$Z_t = \int_0^t g_H(s, t) dX_s, t \geq 0.$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

(3. 6) $$Z_t = \int_0^t Q_{H,\theta}(s) d < M^H >_s + M_t^H$$

where M^H is the fundamental martingale and the process X admits the representation

(3. 7) $$X_t = \int_0^t \hat{g}_H(s, t) dZ_s.$$

Let P_θ^T be the probability measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when θ is the true parameter. Following Theorem 2.1, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

(3. 8) $$\frac{dP_\theta^T}{dP_0^T} = \exp[\int_0^T Q_{H,\theta}(s) dZ_s - \frac{1}{2} \int_0^T Q_{H,\theta}^2(s) d < M^H >_s].$$

We now consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ or equivalently $\{Z_t, 0 \leq t \leq T\}$ and study the asymptotic properties of such estimators as $T \to \infty$.

Let $\{\alpha(t), t \geq 0\}$ be a stochastic process such that the function $\alpha(t)$ is $\mathcal{F}_t$-measurable. An example of such a process is $\alpha(t) = K(t, \tilde{X}(t))$, where $\tilde{X}(t) = \{X(s), 0 \leq s \leq t\}$ and $K(.,.)$ is a real-valued measurable function defined on $R_+ \times R$. Suppose that

$$\int_0^T E[(\alpha(t))^2]d < M^H >_t < \infty.$$

This condition implies that the stochastic integral

$$\int_0^T \alpha(t) dM_t^H$$

exists as a stochastic integral with respect to the martingale $\{M_t^H, \mathcal{F}_t, t \geq 0\}$. In particular

$$E(\int_0^T \alpha(t) dM_t^H) = 0.$$

Observing that

$$dZ_t = dM_t^H + Q_{H,\theta}(t)d < M^H >_t, t \geq 0$$

from (3.6), we can rewrite the above equation in the form

$$E(\int_0^T \alpha(t)(dZ_t - Q_{H,\theta}(t)d < M^H >_t) = 0$$

or equivalently

$$E(\int_0^T \alpha(t)(dZ_t - (J_1(t) + \theta J_2(t))d < M^H >_t)) = 0 \qquad (3.\ 9)$$

where

(3. 10)

$$\begin{aligned} Q_{H,\theta}(t) &= \frac{d}{d < M^H >_t} \int_0^t g_H(s,t) C(\theta, s) ds \\ &= \frac{d}{d < M^H >_t} \int_0^t g_H(t,s) a(s, X(s)) ds + \theta \frac{d}{d < M^H >_t} \int_0^t g_H(t,s) b(s, X(s)) ds \\ &= J_1(t) + \theta J_2(t). \text{ (say).} \end{aligned}$$

A sample analogue of the equation (3.9) is

$$\int_0^T \alpha(t)(dZ_t - (J_1(t) + \theta J_2(t))d < M^H >_t) = 0 \qquad (3.\ 11)$$

which motivates the instrumental variable estimator defined below.

Definition 3.1: Corresponding to the $\mathcal{F}_t$-adapted instrument process $\{\alpha(t), t \geq 0\}$, the *instrumental variable estimator* (IVE) of θ is defined by

$$\tilde{\theta}_T = \frac{\int_0^T \alpha(t)(dZ_t - J_1(t)d < M^H >_t)}{\int_0^T \alpha(t) J_2(t) d < M^H >_t}.$$

Choosing the process $\{\alpha(t), t \geq 0\}$ suitably, we can obtain a class of instrumental variable estimators (IVE) for θ. In analogy with the least squares estimation, we can choose $\alpha(t) = J_2(t)$ as defined above and the corresponding IVE may be called a *least squares estimator* (LSE). In fact, it is the maximum likelihood estimator (MLE) (cf. Prakasa Rao (2003)). In the following discussion, we will choose $\alpha(t) = K(t, \tilde{X}(t))$ where $K(.,.)$ is a real-valued measurable function defined on $R_+ \times R$.

Suppose θ_0 is the true value of the parameter θ. It is easy to check that

$$\tilde{\theta}_T - \theta_0 = \frac{\int_0^T K(t, \tilde{X}(t)) dM_t^H}{\int_0^T K(t, \tilde{X}(t)) J_2(t) d < M^H >_t} \tag{3. 12}$$

using the fact that

$$dZ_t = (J_1(t) + \theta_0 J_2(t)) d < M^H >_t + dM_t^H. \tag{3. 13}$$

We now discuss the problem of instrumental variable estimation of the parameter θ on the basis of the observation of the process X or equivalently the process Z on the interval $[0, T]$.

The equation (3.12) can be written in the form

$$\tilde{\theta}_T - \theta_0 = \frac{\int_0^T K(t, \tilde{X}(t)) dM_t^H}{\int_0^T K(t, \tilde{X}(t))^2 d < M^H >_t} \frac{\int_0^T K(t, \tilde{X}(t))^2 d < M^H >_t}{\int_0^T K(t, \tilde{X}(t)) J_2(t) d < M^H >_t}. \tag{3. 14}$$

Strong Consistency:

Theorem 3.1: The instrumental variable estimator $\tilde{\theta}_T$ is strongly consistent, that is,

$$\tilde{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty \tag{3. 15}$$

provided

$$(i) \int_0^T K(t, \tilde{X}(t))^2 d < M^H >_t \to \infty \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty \tag{3. 16}$$

and

$$(ii) \limsup_{T \to \infty} \left| \frac{\int_0^T K(t, \tilde{X}(t))^2 d < M^H >_t}{\int_0^T K(t, \tilde{X}(t)) J_2(t) d < M^H >_t} \right| < \infty \text{ a.s. } [P_{\theta_0}]. \tag{3. 17}$$

Proof: This theorem follows by observing that the process

$$R_T \equiv \int_0^T K(t, \tilde{X}(t)) dM_t^H, t \geq 0 \tag{3. 18}$$

is a local martingale with the quadratic variation process

(3. 19) $$< R >_T = \int_0^T K(t, \tilde{X}(t))^2(t) d < M^H >_t$$

and applying the Strong law of large numbers (cf. Liptser (1980); Prakasa Rao (1999b), p. 61) under the conditions (i) and (ii) stated above.

Remarks: For the case fractional Ornstein-Uhlenbeck type process driven by a mfBm defined by the equation (1.1), investigated in Chigansky and Kleptsyna (2016), it can be checked that the condition stated in the equation (i) holds when $K(t, \tilde{X}(t)) = J_2(t)$ and hence the maximum likelihood estimator which is also the least squares estimator is strongly consistent as $T \to \infty$.

Limiting distribution:

We now discuss the limiting distribution of the IVE $\tilde{\theta}_T$ as $T \to \infty$. Let

(3. 20) $$\beta_T = \frac{\int_0^T K(t, \tilde{X}(t))^2 d < M^H >_t}{\int_0^T K(t, \tilde{X}(t)) J_2(t) d < M^H >_t}.$$

It is easy to see that

(3. 21) $$\tilde{\theta}_T - \theta_0 = \frac{R_T}{< R >_T} \beta_T.$$

Theorem 3.2: Assume that the functions $b(t,s)$ is such that the process $\{R_t, t \geq 0\}$ is a local continuous martingale and that there exists a process $\{\gamma_t, t \geq 0\}$ such that γ_t is $\mathcal{F}_t$-adapted and

(3. 22) $$\gamma_T^2 < R >_T = \gamma_T^2 \int_0^T K(t, \tilde{X}(t))^2(t) d < M^H >_t \to \eta^2 \text{ in probability as } T \to \infty$$

where $\gamma_T^2 \to 0$ a.s. [P]as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

(3. 23) $$(\gamma_T R_T, \gamma_T^2 < R >_T) \to (\eta Z, \eta^2) \text{ in law as } T \to \infty$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Proof: This theorem follows as a consequence of the Central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao (1999b), p. 65).

Observe that

(3. 24) $$\beta_T^{-1} \gamma_T^{-1} (\tilde{\theta}_T - \theta_0) = \frac{\gamma_T R_T}{\gamma_T^2 < R >_T}.$$

Applying Theorem 3.2, we obtain the following result.

Theorem 3.3: Suppose the conditions stated in the Theorem 3.2 hold. Then

(3. 25) $$(\beta_T\gamma_T)^{-1}(\tilde{\theta}_T-\theta_0)\rightarrow\frac{Z}{\eta} \text{ in law as } T\rightarrow\infty$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: (i)If the random variable η is a constant with probability one, then the limiting distribution of the normalized IVE with random norming is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

(ii) Note that the IVE is not necessarily asymptotically efficient. It is not asymptotically efficient even when the the random variable η is a constant. It is asymptotically efficient in this case if $K(t,\tilde{X}(t))=J_2(t)$ where the process $J_2(t)$ is as defined by the equation (3.10). Observe that the IVE reduces to the MLE in case $K(t,\tilde{X}(t))=J_2(t)$.

4 Berry-Esseen type bound for IVE

Hereafter we assume that the random variable η in (3.22) is a positive constant with probability one. Hence

(4. 1) $$(\beta_T\gamma_T)^{-1}(\tilde{\theta}_T-\theta_0)\rightarrow N(0,\eta^{-2}) \text{ in law as } T\rightarrow\infty$$

where $N(0,\eta^{-2})$ denotes the Gaussian distribution with mean zero and variance η^{-2}. We will now study the rate of convergence of the asymptotic distribution of the IVE in (4.1).

Suppose there exist non-random positive functions δ_T decreasing to zero and ε_T decreasing to zero such that

(4. 2) $$\delta_T^{-1}\varepsilon^2(T)\rightarrow\infty \text{ as } T\rightarrow\infty$$

and

(4. 3) $$\sup_{\theta\in\Theta}P_\theta^T[|\delta_T<R>_T-1|\geq\varepsilon_T]=O(\varepsilon_T^{1/2})$$

where the process $\{R_t,t\geq 0\}$ is as defined in (3.18). Note that the process $\{R_t,t\geq 0\}$ is a locally square integrable continuous martingale. From the results on the representation of locally square integrable continuous martingales (cf. Ikeda and Watanabe (1981), Chapter

II, Theorem 7.2), it follows that there exists a standard Wiener process $\{B(t), t \geq 0\}$ adapted to $(\mathcal{F}_t)$ such that $R_t = B(< R >_T), t \geq 0$. In particular

$$R_T \delta_T^{1/2} = B(< R >_T \delta_T) \text{ a.s. } [P_{\theta_0}] \tag{4. 4}$$

for all $T \geq 0$.

We use the following lemmas in the sequel.

Lemma 4.1: Let $(\Omega, \mathcal{F}, P)$ be a probability space and f and g be $\mathcal{F}$-measurable functions. Then, for any $\varepsilon > 0$,

$$\begin{aligned} &\sup_x |P(\omega : \frac{f(\omega)}{g(\omega)} \leq x) - \Phi(x)| \\ &\leq \sup_y |P(\omega : f(\omega) \leq y) - \Phi(x)| + P(\omega : |g(\omega) - 1| > \varepsilon) + \varepsilon \end{aligned} \tag{4. 5}$$

where $\Phi(x)$ is the distribution function of the standard Gaussian distribution.

Proof: See Michel and Pfanzagl (1971).

Lemma 4.2: Let $\{B(t), t \geq 0\}$ be a standard Wiener process and V be a nonnegative random variable. Then, for every $x \in R$ and $\varepsilon > 0$,

$$|P(B(V) \leq x) - \Phi(x)| \leq (2\varepsilon)^{1/2} + P(|V - 1| > \varepsilon). \tag{4. 6}$$

Proof: See Hall and Heyde (1980), p.85.

Let us fix $\theta \in \Theta$. It is clear from the earlier remarks that

$$R_T =< R >_T \beta_T^{-1}(\tilde{\theta}_T - \theta) \tag{4. 7}$$

under P_θ measure. Then it follows, from the Lemmas 4.1 and 4.2, that

$$\begin{aligned} &P_\theta[\delta_T^{-1/2} \beta_T^{-1}(\hat{\theta}_T - \theta_0) \leq x] - \Phi(x)| \\ &= |P_\theta[\frac{R_T}{< R >_T} \delta_T^{-1/2} \leq x] - \Phi(x)| \\ &= |P_\theta[\frac{R_T / \delta_T^{-1/2}}{< R >_T / \delta_T^{-1}} \leq x] - \Phi(x)| \\ &\leq \sup_x |P_\theta[R_T \delta_T^{1/2} \leq x] - \Phi(x)| \\ &\quad + P_\theta[|\delta_T < R >_T -1| \geq \varepsilon_T] + \varepsilon_T \\ &= \sup_y |P(B(< R >_T \delta_T) \leq y) - \Phi(y)| + P_\theta[|\delta_T < R >_T -1| \geq \varepsilon_T] + \varepsilon_T \\ &\leq (2\varepsilon_T)^{1/2} + 2P_\theta[|\delta_T < R >_T -1| \geq \varepsilon_T] + \varepsilon_T. \end{aligned} \tag{4. 8}$$

It is clear that the bound obtained above is of the order $O(\varepsilon_T^{1/2})$ under the condition (4.3) and it is uniform in $\theta \in \Theta$. Hence we have the following result.

Theorem 4.3: Under the conditions (4.2) and (4.3),

$$\begin{aligned}(4.\ 9) \qquad & \sup_{\theta\in\Theta}\sup_{x\in R}|P_\theta[\delta_T^{-1/2}\beta_T^{-1}(\tilde{\theta}_T-\theta)\le x]-\Phi(x)| \\ & \le (2\varepsilon_T)^{1/2}+2P_\theta[|\delta_T < R >_T -1|\ge \varepsilon_T]+\varepsilon_T = O(\varepsilon_T^{1/2}).\end{aligned}$$

As a consequence of this result, we have the following theorem giving the rate of convergence of the IVE $\tilde{\theta}_T$.

Theorem 4.4: Suppose the conditions (4.2) and (4.3) hold. Then there exists a constant $c > 0$ such that for every $d > 0$,

$$(4.\ 10) \qquad \sup_{\theta\in\Theta} P_\theta[\beta_T^{-1}|\tilde{\theta}_T-\theta|\ge d] \le c\varepsilon_T^{1/2}+2P_\theta[|\delta_T < R >_T -1|\ge\varepsilon_T] = O(\varepsilon_T^{1/2}).$$

Proof: Observe that

$$\begin{aligned}(4.\ 11) \qquad & \sup_{\theta\in\Theta} P_\theta[\beta_T^{-1}|\tilde{\theta}_T-\theta|\ge d] \\ & \le \sup_{\theta\in\Theta}|P_\theta[\delta_T^{-1/2}\beta_T^{-1}(\tilde{\theta}_T-\theta)\ge d\delta_T^{-1/2}]-2(1-\Phi(d\delta_T^{-1/2}))| \\ & \quad +2(1-\Phi(d\delta_T^{-1/2})) \\ & \le (2\varepsilon_T)^{1/2}+2\sup_{\theta\in\Theta}P_\theta[|\delta_T < R >_T -1|\ge\varepsilon_T]+\varepsilon_T \\ & \quad +2d^{-1/2}\delta_T^{1/2}(2\pi)^{-1/2}\exp[-\frac{1}{2}\delta_T^{-1}d^2]\end{aligned}$$

by Theorem 4.3 and the inequality

$$(4.\ 12) \qquad 1-\Phi(x) < \frac{1}{x\sqrt{2\pi}}\exp[-\frac{1}{2}x^2]$$

for all $x > 0$ (cf. Feller (1968), p.175). Since

$$\delta_T^{-1}\varepsilon^2(T)\to\infty \text{ as } T\to\infty$$

by the condition (4.2), it follows that

$$(4.\ 13) \qquad \sup_{\theta\in\Theta}P_\theta[\beta_T^{-1}|\tilde{\theta}_T-\theta|\ge d] \le c\varepsilon_T^{1/2}+2\sup_{\theta\in\Theta}P_\theta[|\delta_T < R >_T -1|\ge\varepsilon_T]$$

for some constant $c > 0$ and the last term is of the order $O(\varepsilon_T^{1/2})$ by the condition (4.3). This proves Theorem 4.4.

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