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Conditions for Singularity for Measures generated by Two Fractional Psuedo-Diffusion Processes

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Abstract: We derive sufficient conditions under which the probability measures generated by two fractional psuedo-diffusion processes are singular with respect to each other.

Key words: Singularity ; Fractional psuedo-diffusion process; Fractional Brownian motion; Sub-fractional Brownian motion; Brownian motion; Baxter theorem; Hurst index.

Mathematics Subject Classification: Primary 60G22; Secondary 60G30.

1 Introduction

Berzin et al. [4] introduced the class of fractional psuedo-diffusion proceseses and studied point estimation and functional estimation for the local variance function for such processes. In our earlier papers (cf. Prakasa Rao [19,21]), we have investigated sufficient conditions under which probability measures generated by two fractional Brownian motions or two sub-fractional Brownian motions are singular with respect to each other. Our aim in this paper is to obtan similar results for fractional psuedo-diffusion processes introduced by Berzin et al. [4] as application of Baxter-type theorems developed in Berzin et al. [4]. The following theorem is due to Baxter [1] (cf. Rao [22], p.249) for Gaussian processes.

Theorem 1.1: Suppose a process $\{X(t), a \leq t \leq b\}$ is a Gaussian process with the mean function $m(t)$ and cthe ovariance function $r(s,t)$. Let $T = [a,b]$. Suppose the function $m(.)$ is differentiable and the function $r(s,t)$ is continuous on the square $T \times T$ with uniformly bounded (mixed) second derivatives just off the diagonal of the square $T \times T$. Let

$$f_r(t) = D^-(t) - D^+(t)$$

where

$$D^+(t) = \lim_{s \to t^+} \frac{r(t,t) - r(s,t)}{t - s}$$

and

$$D^-(t) = \lim_{s \to t^-} \frac{r(t,t) - r(s,t)}{t - s}.$$

Then

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}[X(a+\frac{k(b-a)}{2^n})-X(a+\frac{(k-1)(b-a)}{2^n})]^2=\int_a^b f_r(s)ds \text{ a.s.}$$

As a special case of Theorem 1.1, we have the following result, due to Levy, for the standard Brownian motion.

Theorem 1.2: Consider the standard Brownian motion $\{W(t), 0\leq t\leq T\}$. Then

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}[W(\frac{kT}{2^n})-W(\frac{(k-1)T}{2^n})]^2=T \quad \text{a.s.}$$

2 Baxter-type theorem for fractional Brownian motion

Fractional Brownian motion (fBM) and its properties are described in Mishura [17] and Prakasa Rao [20]. In a paper on estimation of the Hurst index for fBm, Kurchenko [14] derived a Baxter-type theorem for the fractional Brownian motion based on the second order increments of the process.

Let $f:(a,b)\to R$ be a function and let k be a positive integer. Let $\Delta_h^{(k)}f(t)$ denote the increment of k-th order of the function f on an interval $[t,t+h]\subset(a,b)$ as given, namely,

$$\Delta_h^{(k)}f(t)=\sum_{i=0}^{k}(-1)^i {}^kC_i f(t+\frac{i}{k}h).$$

For any $m\geq 0$, positive integer $k\geq 1$ and $0<H<1$, define

$$V_k(m,H)=\frac{1}{2}\sum_{i,j=0}^{k}(-1)^{i+j+1}\,{}^kC_i\,{}^kC_j|m+\frac{i-j}{k}|^{2H}.$$

It can be checked that $V_2(0,H)=2^{2-2H}-1$ and

$$\Delta_1^{(2)}f(t)=f(t)-2f(t+\frac{1}{2})+f(t+1).$$

Kurchenko [14] proved the following Baxter-type theorem based on the second order increments for a fBm among several other results.

Theorem 2.1 : Let $\{W_H(t), t \geq 0\}$ be a standard fractional Brownian motion with Hurst index $H \in (0,1)$. Then, with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} (\Delta_1^{(2)} W_H(m))^2 = V_2(0,H) \text{ a.s.}$$

In other words

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} [W_H(m) - 2W_H(m+\frac{1}{2}) + W_H(m+1)]^2 = V_2(0,H) \text{ a.s.}$$

for any fBm with the Hurst index $H \in (0,1)$.

Decreusefond and Ustunel [8] obtained more general results on the p-th variation of a fractional Brownian motion. Application of fBm for stochastic modeling for long range dependent phenomena are discussed in Doukhan et al. [10].

It is well known that if P and Q are probability measures generated by two Gaussian processes, then these measures are either equivalent or singular with respect to each other (cf. Feldman [11], Hajek [12]). For a proof, see Rao [22], p. 226.

Let $\{W_{H_i}(t), t \geq 0\}, i = 1,2$ be two fractional Brownian motions with the Hurst indices $H_1 \neq H_2$. From the result stated above, it follows that the probability measures generated by these processes are either equivalent or singular with respect to each other. We have proved that they are singular with respect to each other in Prakasa Rao [19] (cf. Prakasa Rao [20]) if $H_1 \neq H_2$. .

Theorem 2.2: Let $\{W_{H_i}(t), t \geq 0\}, i = 1,2$ be two fractional Brownian motions with Hurst indices $H_1 \neq H_2$. Let P_i be the probability measure generated by the process $\{W_{H_i}(t, t \geq 0\}$ for $i = 1,2$. Then the probability measures P_1 and P_2 are singular with respect to each other.

3 Baxter-type theorem for sub-fractional Brownian motion

A Gaussian stochastic process $S^H = \{S^H(t), 0 \leq t < \infty\}$ with $S^H(0) = 0$, zero mean and with the covariance function

$$r(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}]$$

with $0 < H < 1$ is called a *sub-fractional Brownian motion* with Hurst index H. This process is a standard Brownian motion if $H = \frac{1}{2}$. Some probabilistic properties of sub-fractional

Brownian motion are discussed in Bojodecki et al. [5,6] and Tudor [23]. A sub-fractional Brownian motion is neither a semimartingale nor a Markov process. It can be shown that

$$\min[(2-2^{2H-1}),1]|t-s|^{2H} \leq E|S(t)-S(s)|^{2H} \leq \max[(2-2^{2H-1}),1]|t-s|^{2H}.$$

Let

$$V_n(S^H) = \sum_{k=0}^{n-1}[S^H(\frac{k+2}{n}) + S^H(\frac{k}{n}) - 2S^H(\frac{k+1}{n})]^2, n \geq 1. \tag{3. 1}$$

Liu and Yan [16] proved the following theorem as an application of results in Begyn [2] on limit theorems for quadratic variations of Gaussian processes.

Theorem 3.1: Let S^H be a sub-fractional Brownian motion with the Hurst index H. Define the the sequence of random variables $V_n(S^H)$ as in (3.1). Then

$$\lim_{n\to\infty} n^{2H-1}V_n(S^H) = 4 - 2^{2H} \text{ a.s.} \tag{3. 2}$$

As a consequence of Theorem 3.1, we proved that two probability measures generated by sub-fractional Brownian motions with different Hurst indices are singular with respect to each other in Prakasa Rao [20].

Theorem 3.2: Let $\{S^{H_i}(t), t \geq 0\}, i = 1, 2$ be two sub-fractional Brownian motions with Hurst indices $H_1 \neq H_2$. Let P_i be the probability measure generated by the process $\{S^{H_i}(t), t \geq 0\}$ for $i = 1, 2$. Then the probability measures P_1 and P_2 are singular with respect to each other.

4 Fractional Pseudo-Diffusion Processes

In their recent work, Berzin et al. [4] introduced a class of processes known as fractional psuedo-diffusion models. These are processes $\{X(t), t \geq 0\}$ satisfying the stochastic differential equation (SDE)

$$dX(t) = \mu(t)dt + \sigma(t)dY(t), t \geq 0 \tag{4. 1}$$

where the functions $\sigma(.)$ and $\mu(.)$ are deterministic or random functions and the process Y is a Gaussian process which can be considered as a smooth perturbation of a fractional Brownian motion. The process X can be considered as a generalization of a fractional diffusion process (cf. Prakasa Rao [20]). Following Berzin et al. [4], we will consider three classes of fractional pseudo-diffusion processes:

(i) Fractional pseudo-diffusion with deterministic coefficients $\sigma(t)$ and $\mu(t)$ (fDdc);
(ii) Generalized fractional Ornstein-Uhlenbeck process (gfOUp);
(iii) Fractional pseudo-diffusion with random coefficients $\sigma(t) = \sigma(Y(t))$ and $\mu(t) = \mu(X(t))$ (fDrc).

Fractional pseudo-diffusion with deterministic coefficients $\sigma(t)$ and $\mu(t)$ (fDdc):

Berzin et al. [4] studied estimation of the local variance function $\sigma(.)$ based on a discrete set of observations on the process X observed at times when the norm of the mesh of the times of observation tends to zero. We now define the process Y which is the driving force in the SDE given by the equation (4.1). The process Y is a centered Gaussian process given by the equation

$$Y(t) = \int_{-\infty}^{\infty} [e^{it\lambda} - 1]\sqrt{f(\lambda)}dW(\lambda), t \geq 0 \tag{4. 2}$$

where W is the standard Brownian motion and the function f is of the form

$$f(\lambda) = \frac{1}{2\pi}|\lambda|^{-2H-1} + G(\lambda) \tag{4. 3}$$

with G an even positive integrable function. Note that the process Y is a Gaussian process. An example of such a process Y is

$$Y(t) = W_H(t) + Z(t), t \geq 0 \tag{4. 4}$$

where W_H is a fBm with Hurst index H and the covariance function given by

$$Cov(W_H(t), W_h(s)) = \frac{1}{2}v_H^2[|t|^{2H} + |s|^{2H} - |t-s|^{2H}], v_H^2 = [\Gamma(2H+1)\ \sin(\pi H)]^{-1}$$

and Z is an independent stationary Gaussian process with $Z(0) = 0$ and with the covariance function $\hat{G}$, the Fourier transform of the function G. From the representation given in Hunt [13], it can be seen that

$$W_H(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{i\lambda t} - 1]|\lambda|^{-H-\frac{1}{2}}dW(\lambda), t \geq 0 \tag{4. 5}$$

which implies that the fBm is a particular case of the process Y defined by (4.2) with $G \equiv 0$. Suppose the function $\sigma(.)$ belongs to C^1 on the interval $[0, 1]$ and the function $\mu(.)$ is Lipschitz on $[0,1]$. Then the SDE (4.1) has a unique solution given by

$$(4.\ 6)\ X_\mu(t) = X_\mu(0) + Y(t)\sigma(t) - Y(0)\sigma(0) - \int_0^t \sigma'(u)Y(u)du + \int_0^t \mu(u)du, 0 \leq t \leq 1$$

from the results in Nualart and Rascanu [18]. Here $\sigma'(u)$ denotes the derivative of $\sigma(u)$. We call such a process X as a *fractional pseudo-diffusion with deterministic coefficients (fDdc).* From the representation of the process X_μ, it follows that the process $\{X_\mu, t \geq 0\}$ is a Gaussian process provided the random variable $X_\mu(0)$ is a constant or a random variable with a Gaussian distribution independent of the process $\{Y(t), t \geq 0\}$.

Generalized fractional Ornstein-Uhlenbeck process (gfOUp):

Suppose the function $\sigma(.)$ is positive and belongs to C^1 on the interval $[0,1]$. Let X_λ be a stochastic process satisfying the stochastic differential equation

$$(4.\ 7) \qquad dX_\lambda(t) = -\lambda X_\lambda(t)dt + \sigma(t)dY(t), 0 \leq t \leq 1, \lambda > 0$$

where the process Y is as defined by (4.2). The solution of this equation is given by

$$(4.\ 8) \qquad X_\lambda(t) = e^{-\lambda t}(X_\lambda(0) + \int_0^t \sigma(u)e^{\lambda u}dY(u)).$$

From the representation of the process X_λ, it follows that the process is a Gaussian process provided the initial random variable $X_\lambda(0)$ is a constant or a random variable with a Gaussian distribution independent of the process $\{Y(t), t \geq 0\}$.

If $\sigma(.)$ is a constant and Y is a fBm (which is the case if the function $G \equiv 0$), then the process X_λ is a fractional Ornstein-Uhlenbeck process (see Cheridito, Kawaguchi and Maejima [7]; Prakasa Rao [20]).

Let us also consider a more general model

$$(4.\ 9) \qquad dX_\lambda(t) = \mu(X_\lambda(t))dt + \sigma(t)dW_H(t), 0 \leq t \leq 1$$

where W_H is the fBm with the Hurst index $H > \frac{1}{2}$, the function $\sigma(.)$ is deterministic and belongs to C^1 on the interval $[0,1]$, and the function $\mu(.)$ is Lipschitz on R. Then there exists a unique solution X_λ for the SDE given by (4.9) and it is given by

$$(4.\ 10)\ X_\lambda(t) = X_\lambda(0) + W_H(t)\sigma(t) - \int_0^t \sigma'(u)W_H(u)du + \int_0^t \mu(X_\lambda(u))du, 0 \leq t \leq 1$$

and, with probability one, the process X_λ has $(H-\delta)$-Holder continuous trajectories on $[0,1]$ for any $0<\delta<H$ (Nualart and Rascanu [18]). We will consider both the models described above as *generalized fractional Ornstein-Uhlenbeck processes (gfOUp).* If $\mu(x) = -\lambda x$, then the process X_λ reduces to the fractional Ornstien-Uhlenbeck process and it is a Gaussian process if $X_\lambda(0)$ is a constant or a Gaussian random variable independent of the process Y.

Fractional pseudo-diffusion with random coefficients (fDrc):

Consider the process Y as defined by (4.2). The process Y has zero quadratic variation for $H > \frac{1}{2}$ by Lemma 7.1 in Berzin et al. [4]. Suppose the function $\sigma(.)$ is positive and belongs to C^1 on R and the function $\mu(.)$ is continuous on R. Consider the fractional psuedo-diffusion model

$$dX(t) = \mu(Y(t))dt + \sigma(Y(t))dY(t), t \geq 0. \tag{4. 11}$$

Here the functions $\mu(Y(t))$ and $\sigma(Y(t))$ are both random. The solution X of the SDE given by the equation (4.11) is

$$X(t) = X(0) + \int_0^{Y(t)} \sigma(u)du + \int_0^t \mu(Y(u))du, t \geq 0 \tag{4. 12}$$

by results in Lin [15]. We call such a process X as a *fractional pseudo-diffusion with random coefficients (fDrc).*

5 Baxter-type theorems for Fractional Pseudo-Diffusion Processes

For any integer $n \geq 2$, let $\Delta_n W_H$ be the second order increment of the process W_H, defined as $\Delta_n W_H(i)$, given by the relation

$$\Delta_n W_H(i) = \frac{n^H}{\eta_{2H}}[W_H(\frac{i+2}{n}) - 2W_H(\frac{i+1}{n}) + W_H(\frac{i}{n})], i = 0, 1, \ldots, n \tag{5. 1}$$

where

$$\eta_{2H}^2 = v_{2H}^2(4 - 2^{2H}) = [\Gamma(2H+1)\sin(\pi H)]^{-1}(4 - 2^{2H}). \tag{5. 2}$$

Baxter-type theorem for fDdc:

Suppose the model (4.1) holds and the driving force is the process Y defined by the equation (4.2). Further suppose the function $\mu(.)$ is Lipschitz on the interval $[0,1]$ and the function $\sigma 9.)$ belongs to C^1 on the interval $[0,1]$ and is strictly positive in the interval $[0,1]$. Suppose the process X_μ, given by the equation (4.6) is observed over the grid $\{\frac{i}{n}, i = 0,1,\ldots,n\}, n \geq 2$. Define

$$\Delta_n X_\mu(i) = \frac{n^H}{\eta_{2H}}[X_\mu(\frac{i+2}{n}) - 2X_\mu(\frac{i+1}{n}) + X_\mu(\frac{i}{n})], i = 0,1,\ldots,n. \quad (5.\ 3)$$

As a special case of Theorem 4.1 in Berzin et al. [4], we obtain the following theorem.

Theorem 5.1: Under the conditions stated above,

$$\frac{1}{n-1}\sum_{i=0}^{n-2}[\Delta_n X_\mu(i)]^2 \to \int_0^1 \sigma^2(u)du \quad \text{a.s. as } n \to \infty. \quad (5.\ 4)$$

We can rewrite the convergence statement stated in Theorem 5.1 in the form

$$\frac{n^{2H}}{n-1}\sum_{i=0}^{n-2}[X_\mu(\frac{i+2}{n}) - 2X_\mu(\frac{i+1}{n}) + X_\mu(\frac{i}{n})]^2 \to \eta_{2H}^2 \int_0^1 \sigma^2(u)du \quad \text{a.s. as } n \to \infty.$$

This result was proved in a more general form for a fBm in Kurchenko [14] and in Berzin et al. [3]. Let $X(\mu,\sigma,G,H)$ denote the fractional psuedo-diffusion process defined by the model (4.1) generated by the process Y with the Hurst index H, a positive even integrable function G and the deterministic functions μ and σ. Suppose the random variable $X_\mu(0)$ is a constant or a Gaussian random variable independent of the process Y. Then the process $X(\mu,\sigma,G,H)$ is a Gaussian process as observed earlier and it obeys the Baxter-type theorem stated in Theorem 5.1. Note that the limit in Theorem 5.1 does not depend on the functions μ and G but depends on the Hurst index H and the function σ and the convergence in Theorem 5.1 holds almost surely. Since the probability measures generated by any two Gaussian processes are either equivalent or singular with respect to each other (cf. Feldman [11], Hajek 12, Rao [22]), as a consequence of Theorem 5.1, it follows that the probability measures generated by any two fractional psuedo-diffusion processes $X(\mu_1,\sigma_1,G_1,H_1)$ and $X(\mu_2,\sigma_2,G_2,H_2)$ are singular with respect to each other if

$$\eta_{2H_1}^2 \int_0^1 \sigma_1^2(u)du \neq \eta_{2H_2}^2 \int_0^1 \sigma_2^2(u)du.$$

In particular, if $\sigma_1 = \sigma_2$, then the probability measures generated by the processes $X(\mu_1,\sigma_1,G_1,H_1)$ and $X(\mu_2,\sigma_2,G_2,H_2)$ are singular with respect to each other if $H_1 \neq H_2$ and if $H_1 = H_2$, then the probability measures generated by the processes $X(\mu_1,\sigma_1,G_1,H_1)$ and $X(\mu_2,\sigma_2,G_2,H_2)$ are singular with respect to each other if $\int_0^1 \sigma_1^2(u)du \neq \int_0^1 \sigma_2^2(u)du$. We have the following result.

Theorem 5.2: Let P_i be the probability measure generated by the fractional-psuedo-diffusion process $X(\mu_i, \sigma_i, G_i, H_i), i = 1, 2$ induced by the process Y given by the equation (4.2) and suppose that the random variable $X_{\mu_i}(0)$ is Gaussian independent of the process Y for $i = 1, 2$. Then the measures P_1 and P_2 are singular with respect to each other if

$$\eta_{2H_1}^2 \int_0^1 \sigma_1^2(u)du \neq \eta_{2H_2}^2 \int_0^1 \sigma_2^2(u)du$$

where

$$\eta_{2H}^2 = [\Gamma(2H+1)\sin(\pi H)]^{-1}(4 - 2^{2H}).$$

Baxter-type theorem for gfOUp:

Suppose the model (4.7) or (4.9) holds with the driving force Y defined by (4.2), the function $\sigma(.)$ belongs to C^2 on the interval $[0, 1]$ and the function $\mu(.)$ is Lipschitz on R if model (4.9) holds.Suppose the process X_λ is observed over the grid $\{\frac{i}{n}, i = 0, 1, \ldots, n\}, n \geq 2$. Define

(5. 5) $$\Delta_n X_\lambda(i) = \frac{n^H}{\eta_{2H}}[X_\lambda(\frac{i+2}{n}) - 2X_\lambda(\frac{i+1}{n}) + X_\lambda(\frac{i}{n})]$$

As a special case of Theorem 4.3 in Berzin et al. [4], we obtain the following theorem.

Theorem 5.3: Under the conditions stated above,

(5. 6) $$\frac{1}{n-1}\sum_{i=0}^{n-2}[\Delta_n X_\lambda(i)]^2 \to \int_0^1 \sigma^2(u)du \quad \text{a.s. as } n \to \infty.$$

We can rewrite the convergence statement stated in Theorem 5.3 in the form

$$\frac{n^{2H}}{n-1}\sum_{i=0}^{n-2}[X_\lambda(\frac{i+2}{n}) - 2X_\lambda(\frac{i+1}{n}) + X_\lambda(\frac{i}{n})]^2 \to \eta_{2H}^2 \int_0^1 \sigma^2(u)du \quad \text{a.s. as } n \to \infty.$$

Suppose that the model (4.7) holds with the driving force Y defined by the equation (4.2). Let $X(\lambda, \sigma, G, H)$ denote the generalized fractional Ornstein-Uhlenbeck process generated by the process Y with positive constant λ, Hurst index H, positive even integrable function G and deterministic function $\sigma(.)$. Suppose the random variable $X_\lambda(0)$ is a constant or a Gaussian random variable independent of the process Y. Then the process $X(\lambda, \sigma, G, H)$ is a Gaussian process as observed earlier and it obeys the Baxter-type theorem stated in Theorem 5.3. Note that the limit in Theorem 5.3 does not depend on the functions $\mu(.)$ and G but depends on the Hurst index H and the function $\sigma(.)$ and the convergence in Theorem 5.3 holds almost surely. Since the probability measures generated by any two Gaussian processes are either equivalent or singular with respect to each other (Feldman [11], Hajek [12], Rao[22]), as a consequence of Theorem 5.3, it follows that the probability measures generated by any two generalized fractional Ornstein-Uhlenbeck processes $X(\lambda_1, \sigma_1, G_1, H_1)$ and $X(\lambda_2, \sigma_2, G_2, H_2)$, generated by the SDE (4.7), are singular with respect to each other if

$$\eta^2_{2H_1} \int_0^1 \sigma_1^2(u)du \neq \eta^2_{2H_2} \int_0^1 \sigma_2^2(u)du.$$

In particular, if $\sigma_1 = \sigma_2$, then the probability measures generated by the processes $X(\mu_1, \sigma_1, G_1, H_1)$ and $X(\mu_2, \sigma_2, G_2, H_2)$ are singular with respect to each other if $H_1 \neq H_2$ and if $H_1 = H_2$, then the probability measures generated by the processes $X(\mu_1, \sigma_1, G_1, H_1)$ and $X(\mu_2, \sigma_2, G_2, H_2)$ are singular with respect to each other if $\int_0^1 \sigma_1^2(u)du \neq \int_0^1 \sigma_1^2(u)du$. We have the following result.

Theorem 5.4: Let P_i be the probability measure generated by the fractional-psuedo-diffusion process $X(\lambda_i, \sigma_i, G_i, H_i), i = 1, 2$ defined by the model (4.7) and driven by the process Y given by the equation (2.2). Suppose that the random variable $X\lambda_i(0), i = 1, 2$ is Gaussian independent of the process Y for $i = 1, 2$. Then the probability measures P_1 and P_2 are singular with respect to each other if

$$\eta^2_{2H_1} \int_0^1 \sigma_1^2(u)du \neq \eta^2_{2H_2} \int_0^1 \sigma_2^2(u)du$$

where

$$\eta^2_{2H} = [\Gamma(2H+1)\sin(\pi H)]^{-1}(4 - 2^{2H}).$$

Baxter-type theorem for fDrc:

Suppose the model (4.11) holds with the driving force Y defined by the equation (4.2), the function σ belongs to C^1 on R and the function μ is locally Lipschitz on R. Further suppose that the index $H > \frac{1}{2}$. Suppose the process X is observed over the grid $\{\frac{i}{n}, i = 0, 1, \ldots, n\}, n \geq 2$. Define

$$\Delta_n X(i) = \frac{n^H}{\eta_{2H}}[X(\frac{i+2}{n}) - 2X(\frac{i+1}{n}) + X(\frac{i}{n})], i = 0, 1, \ldots, n. \tag{5. 7}$$

As a special case of Theorem 4.5 in Berzin et al. [4], we obtain the following theorem.

Theorem 5.5: Under the conditions stated above,

$$\frac{1}{n-1}\sum_{i=0}^{n-2}[\Delta_n X(i)]^2 \to \int_0^1 \sigma^2(Y(u))du \quad \text{a.s. as } n \to \infty. \tag{5. 8}$$

We can rewrite the convergence statement stated in Theorem 5.5 in the form

$$\frac{n^{2H}}{n-1}\sum_{i=0}^{n-2}[X(\frac{i+2}{n})-2X(\frac{i+1}{n})+X(\frac{i}{n})]^2 \to \eta_{2H}^2\int_0^1 \sigma^2(u)du \quad \text{a.s. as } n\to\infty.$$

Note that the solution of the model defined by (4.11) is given by

$$X(t) = X(0) + \int_0^{Y(t)} \sigma(u)du + \int_0^t \mu(Y(u))du, t \geq 0. \tag{5. 9}$$

It is not possible to claim that the process X is Gaussian even in case μ is linear and the initial random variable $X(0)$ is Gaussian independent of the process Y. Hence no conclusion on the singularity of measures generated by the process X for different values of the parameters G, H, μ, σ can be given in this case as a consequence of the Baxter-type theorem 5.3 for fDrc.

Remarks: (i) With the use of Baxter-type theorems discussed above, one can give sufficient conditions for singularity of probability measures generated by fractional psuedo-diffusion processes even when the driving force Y is of the form defined either by the equation (2.5) or by the equation (2.6) in Berzin et al. [4]. We omit the details.

(ii) Since a fBm is a special case of the driving force process Y defined by the equation (4.2) (with $G \equiv 0$), it follows that all the results obtained above will continue to hold for probability measures generated by the solutions of SDE given by the model (4.1) with a fBm as the driving force.

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