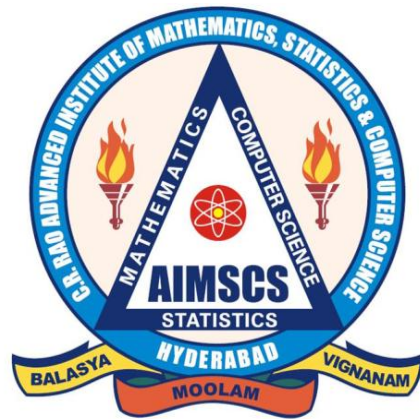


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Author (s): M.N. Mishra and B.L.S. Prakasa Rao

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**Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org**

ESTIMATION OF DRIFT PARAMETER AND CHANGE-POINT FOR SWITCHING FRACTIONAL DIFFUSION PROCESSES

M.N. Mishra and B.L.S. Prakasa Rao

Institute of Mathematics and Applications, Bhubaneswar, India

and

CR Rao Advanced Institute of Mathematics, Statistics and

Computer Science, Hyderabad, India

We study the asymptotic distribution of the maximum likelihood estimator for the drift parameter and the change point for fractional diffusion processes as the noise intensity tends to zero.

Keywords: Change point; fractional Brownian motion; fractional diffusion process; estimation.

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1 Introduction

Change-point problems or disorder problems have been of interest to statisticians for their applications and for probabilists for their challenging problems. Recent applications of change-point methods include finance, statistical image processing and edge detection in noisy images which can be considered as a multidimensional change-point and boundary detection problem. Estimation of change-points in economic models such as split or two-phase regression and changes in hazard or failure rates in modelling life times after bone-marrow transplantation of leukemia patients is of practical interest. A study of change-point problems and their applications are discussed in the monograph on change-point problems edited by Carlstein et al. [3]. Csorgo and Horvath [5] discuss limit theorems in change point analysis. Deshayes and Picard [7] study asymptotic distributions of tests and estimators for change point in the classical statistical model of independent observations (cf. Prakasa Rao [28]). The problem of estimation of both the change point and parameters in the drift and diffusion has been considered recently by many authors in continuous as well as discrete time. The disorder problem for diffusion type processes, that is, processes driven by Wiener process, is investigated in Kutoyants [15], Kutoyants [16] and more recently in Kutoyants [17]. Kutoyants [16] considered the problem of simultaneous estimation of the trend parameter and the

change point for diffusion type processes. Prakasa Rao [29] gives a comprehensive survey on problems of estimation for diffusion type processes observed over in continuous time or over discrete time. For some recent work on the change point problems for diffusion processes, see Lee et al. [19], Song and Lee [33], De Gregorio and Iacus [6] and Iacus and Yoshida [10],[11]. Mishra and Prakasa Rao [21] considered the problem of estimation of change point for switching fractional diffusion processes.

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao [29]. There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion (fBm) in view of their applications for modeling time series which are long-range dependent. In a recent paper, Kleptsyna and Le Breton [13] studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$(1. 1) \quad X_t = X_0 + \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \rightarrow \infty$. A survey of results on statistical inference for fractional diffusion processes, that is, processes driven by a fractional Brownian motion, is given in Prakasa Rao [30]. For more recent work on parametric estimation for fractional Ornstein-Uhlenbeck process, see Xiao et al. [34], Hu and Nualart [8] and Hu et al. [9].

Our aim in this paper is to consider estimation of the drift parameter θ and the change point τ for a model of fractional diffusion process with small diffusion coefficient. We consider the model

$$(1. 2) \quad dX_t = S_t(\theta, \tau, X) dt + \epsilon dW_t^H, X_0 = x_0, 0 \leq t \leq T$$

where $\{W_t^H, 0 \leq t \leq T\}$ is the fractional Brownian motion with known Hurst index $H \in [1/2, 1)$, $S_t(\theta, \tau, x) = h_t(\theta, x)$ if $t \in [0, \tau]$ and $S_t(\theta, \tau, x) = g_t(\theta, x)$ if $t \in (\tau, T]$, with $h_t(\theta, x)$ and $g_t(\theta, x)$ known functions. We estimate the drift parameter θ and the change point τ by the maximum likelihood estimator $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ and study its asymptotic properties following

the methods in Ibragimov and Khasminskii [12] and Prakasa Rao [26]. We show that the normalized sequence

$$(\epsilon^{-1}(\hat{\theta}_\epsilon - \theta), \epsilon^{-2}(\hat{\tau}_\epsilon - \tau))$$

has a limiting distribution as $\epsilon \rightarrow 0$. We note that the change point problems belong to a class of non-regular statistical problems in the sense that the rate of convergence of the estimator for the change point is higher than the standard rate of convergence of the maximum likelihood estimator of a smooth parameter in the classical case of independent and identically distributed observations with a density function which is twice differentiable and with finite positive Fisher information. This was earlier observed by Chernoff and Rubin [4], Deshayes and Picard [7] in their study of estimation of the change point and by Prakasa Rao [26] in his study of estimation of the location of the cusp of a continuous density. Observe that the rate of convergence of the estimator $\hat{\tau}_\epsilon$ is ϵ^2 as $\epsilon \rightarrow 0$ and the rate of convergence of the estimator $\hat{\theta}_\epsilon$ is ϵ as $\epsilon \rightarrow 0$.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are (\mathcal{F}_t) -adapted. Further the natural filtration of a process is understood as the P -completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$(2.1) \quad E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s - t|^{2H}], t \geq 0, s \geq 0.$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$(2.2) \quad Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0$$

where $C = \{C(t), t \geq 0\}$ is an (\mathcal{F}_t) -adapted process and $B(t)$ is a non-vanishing non-random function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$(2.3) \quad dY_t = C(t)dt + B(t)dW_t^H, t \geq 0; Y_0 = 0$$

driven by the fractional Brownian motion W^H . The integral

$$(2.4) \quad \int_0^t B(s)dW_s^H$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to a fractional Brownian motion in a natural sense (cf. Norros et al. [23], Alos et al. [1]). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration (\mathcal{Z}_t) of the process Z coincides with the natural filtration (\mathcal{Y}_t) of the process Y (Kleptsyna et al. [14]). Define, for $0 < s < t$,

$$(2.5) \quad k_H = 2H \Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right),$$

$$(2.6) \quad \kappa_H(t, s) = k_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H},$$

$$(2.7) \quad \lambda_H = \frac{2H \Gamma(3-2H)\Gamma\left(H + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2} - H\right)},$$

$$(2.8) \quad w_t^H = \lambda_H^{-1} t^{2-2H},$$

and

$$(2.9) \quad M_t^H = \int_0^t \kappa_H(t, s) dW_s^H, t \geq 0.$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. [23]) and its quadratic variation is given by $\langle M_t^H \rangle = w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H . In fact the stochastic integral

$$(2.10) \quad \int_0^t B(s) dW_s^H$$

can be represented in terms of the stochastic integral with respect to the martingale M^H . For a measurable function f on $[0, T]$, let

$$(2.11) \quad K_H^f(t, s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{1}{2}} dr, 0 \leq s \leq t$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. [32] for sufficient conditions). The following result is due to Kleptsyna et al. [14].

Theorem 2.1: *Let M^H be the fundamental martingale associated with the fractional Brownian motion W^H defined by (2.9). Then*

$$(2.12) \quad \int_0^t f(s) dW_s^H = \int_0^t K_H^f(t, s) dM_s^H, t \in [0, T]$$

P-a.s. whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. [29]) so that

$$(2.13) \quad Q_H(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) \frac{C(s)}{B(s)} ds, t \in [0, T]$$

is well-defined where w^H and κ_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. The following theorem due to Kleptsyna et al. [14] associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration (\mathcal{Z}_t) of Z coincides with the natural filtration (\mathcal{Y}_t) of Y .

Theorem 2.2: *Suppose the sample paths of the process Q_H defined by (2.13) belong P -a.s to $L^2([0, T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0, T])$ be defined by*

$$(2.14) \quad Z_t = \int_0^t \kappa_H(t, s) B^{-1}(s) dY_s$$

where the function $\kappa_H(t, s)$ is as defined in (2.6). Then the following results hold:

(i) *The process Z is an (\mathcal{F}_t) -semimartingale with the decomposition*

$$(2.15) \quad Z_t = \int_0^t Q_H(s) dw_s^H + M_t^H$$

where M^H is the fundamental martingale defined by (2.9),

(ii) *the process Y admits the representation*

$$(2.16) \quad Y_t = \int_0^t K_H^B(t, s) dZ_s$$

where the function K_H^B is as defined in (2.11), and

(iii) *the natural filtrations (\mathcal{Z}_t) and (\mathcal{Y}_t) coincide.*

Kleptsyna et al. [14] derived the following Girsanov-type formula as a consequence of the Theorem 2.2.

Theorem 2.3: *Suppose the assumptions of Theorem 2.2 hold. Define*

$$(2.17) \quad \Lambda_H(T) = \exp\left\{-\int_0^T Q_H(t) dM_t^H - \frac{1}{2} \int_0^T Q_H^2(t) dw_t^H\right\}.$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^ = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined*

by

$$(2. 18) \quad V_t = \int_0^t B(s)dW_s^H, 0 \leq t \leq T$$

under the probability measure P .

3 Assumptions and main result

Let a process $X = \{X_t, 0 \leq t \leq T\}$ be observed over the time interval $[0, \tau]$ and suppose it follows the stochastic integral equation

$$(3. 1) \quad X_t = x_0 + \int_0^t h_u(\theta, X)du + \epsilon \int_0^t dW_u^H, t \in [0, \tau]$$

and suppose the process $X^* = \{X^*(t), \tau < t \leq T\}$ is observed over the time interval $(\tau, T]$ satisfying the stochastic integral equation

$$(3. 2) \quad X_t^* = X_\tau + \int_\tau^t g_u(\theta, X^*)du + \epsilon \int_\tau^t dW_u^H, t \in (\tau, T].$$

where W^H is the fractional Brownian motion with known Hurst parameter $H \in [\frac{1}{2}, 1)$. Let

$$\begin{aligned} S_t(\theta, \tau, x) &= h_t(\theta, x), \quad 0 \leq t \leq \tau \\ &= g_t(\theta, x), \quad \tau < t \leq T. \end{aligned}$$

Suppose that the change point $\tau \in [t_1, t_2] \subset [0, T]$ where t_1 and t_2 are known but arbitrary in the interval $[0, T]$. We assume that the functions $g_t(\theta, \cdot)$ and $h_t(\theta, \cdot)$ are known but the drift parameter $\theta \in \Theta$ compact and the change point τ are unknown. For convenience, we denote the process X^* by X over the interval $(\tau, T]$. It is required to estimate the change point τ and the drift parameter θ from the realization of X over the interval $[0, T]$. Let $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ denote the maximum likelihood estimator (MLE) of (θ, τ) . We are interested in the asymptotic behaviour of the MLE $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ as $\epsilon \rightarrow 0$.

Let $x = \{x(t), 0 \leq t \leq T\}$ with $x(0) = x_0$ be the solution of the ordinary differential equation

$$\begin{aligned} \frac{dx(t)}{dt} &= h_t(\theta, x), 0 \leq t \leq \tau \\ &= g_t(\theta, x), \tau < t \leq T. \end{aligned}$$

We assume that the trend coefficient $S_t(\theta, \tau, X)$ satisfies the following conditions which ensure the existence and the uniqueness of a unique solution of the equation (1.2).

(A.1) There exists a constant $L > 0$ independent of θ such that

$$|S_t(\theta, \tau, X) - S_t(\theta, \tau, Y)| \leq L |X_t - Y_t| + L \int_0^t |X_s - Y_s| ds, 0 \leq t, \tau \leq T, \theta \in \Theta.$$

(A.2) There exists a constant $M > 0$ independent of θ such that

$$|S_t(\theta, \tau, X)| \leq M (1 + |X_t|) + M \int_0^t (1 + |X_s|) ds, t, \tau \in [0, T], \theta \in \Theta.$$

The existence and the uniqueness of the solution of the stochastic differential equation (1.2) follow from the results in Nualart and Rascanu [24].

The general method of obtaining the asymptotic properties of the maximum likelihood estimator (MLE) for the change point τ by Taylor's expansion of the log-likelihood is not applicable in this situation due to non-differentiability of the likelihood ratio with respect to the parameter τ . Therefore we follow the technique used by Prakasa Rao [26], Ibragimov and Khasminskii [12], Kutoyants [15] and others. We prove the weak convergence of the appropriately normalized log-likelihood ratio random field and appeal to the continuous mapping theorem to study the asymptotic behaviour of the MLE.

Let

$$(3.3) \quad A_t(\theta, \tau, x) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) S_s(\theta, \tau, x) ds, 0 \leq t \leq T.$$

Consider the transformed processes

$$(3.4) \quad A_t(\theta, \tau, X) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) S_s(\theta, \tau, X) ds, 0 \leq t \leq T,$$

$$(3.5) \quad Y_t = \int_0^t \kappa_H(t, s) dX(s), 0 \leq t \leq T$$

and the martingale

$$(3.6) \quad M_t^H = \int_0^t \kappa_H(t, s) dW_s^H, 0 \leq t \leq T.$$

Then the process $Y = \{Y_t, 0 \leq t \leq T\}$ defined by (3.5) satisfies the stochastic differential equation

$$(3.7) \quad dY_t = A_t(\theta, \tau, X)dw_t^H + \epsilon dM_t^H, 0 \leq t \leq T$$

where M^H is the fundamental martingale given by (3.6) and Y is a semimartingale (cf. Kleptsyna et al. [14]).

(A.3) Suppose the function $A_t(\theta, \tau, x)$ satisfies the condition that there exists a constant $C > 0$ independent of θ and τ such that

$$|A_t(\theta, \tau, x) - A_t(\theta, \tau, y)| \leq C \sup_{0 \leq s \leq t} |x_s - y_s|, 0 \leq t \leq T, 0 \leq t_1 \leq \tau \leq t_2 \leq T, \theta \in \Theta.$$

Let

$$(3.8) \quad \Delta_t = A_t(\theta + \epsilon u, \tau + \epsilon^2 v, X) - A_t(\theta, \tau, X)$$

and

$$(3.9) \quad \bar{\Delta}_t = A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)$$

for given $u, v \in R$.

(A.4)(i) Let

$$\delta_t = A_t(\theta + \epsilon u_1, \tau + \epsilon^2 v_1, X) - A_t(\theta + \epsilon u_2, \tau + \epsilon^2 v_2, X).$$

Denote $\tau + \epsilon^2 v_1 = \zeta_1, \tau + \epsilon^2 v_2 = \zeta_2, \theta + \epsilon u_1 = \beta_1$ and $\theta + \epsilon u_2 = \beta_2$ for convenience. Suppose there exists a neighbourhood $N_{(\theta, \tau)}$ of (θ, τ) such that

$$(3.10) \quad \sup_{(\beta_1, \zeta_1), (\beta_2, \zeta_2) \in N_{\theta, \tau}} \sup_{0 \leq t \leq T} E_{\beta_1, \zeta_1}(\delta_t^8) < \infty.$$

(A.4)(ii) Suppose that there exists a positive constant $J_{\theta, \tau}$ depending on θ and τ such that

$$\frac{1}{\epsilon^2} \int_{\tau}^{\tau + \epsilon^2 v} \bar{\Delta}_t^2 dw_t^H = v \lambda_H^{-1} J_{\theta, \tau}^2 \tau^{1-2H} + o(1)$$

as $\epsilon \rightarrow 0$.

In addition to the conditions (A.1) to (A.4), we assume that the following condition holds:

(A.5) There exist constants $c > 0$, and $C > 0$ possibly depending on H, T and Θ such that

$$c g(u, v) \leq \epsilon^{-2} \int_0^T \bar{\Delta}_t^2 dw_t^H \leq C g(u, v), \tau \in [t_1, t_2], \theta \in \Theta$$

for some $g(u, v) = k_1|u|^\alpha + k_2|v|^\beta > 0$ for some $k_1 > 0, k_2 > 0, \alpha > 0$ and $\beta > 0$.

The filtrations of the transformed process Y and the process X coincide by Theorem 1 of Kleptsyna et al. [14] and hence the problem of estimation of the parameter (θ, τ) from the process X and the problem of estimation of the bivariate parameter (θ, τ) from the process Y are equivalent. We now consider the problem of estimation of the smooth drift parameter θ and the change point τ based on the observation $\{Y_t, 0 \leq t \leq T\}$ by the method of maximum likelihood. Let $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ denote the maximum likelihood estimator of (θ, τ) based on the observation $\{Y_t, 0 \leq t \leq T\}$. Let

$$(3.11) \quad J_{\theta, \tau}^* = J_{\theta, \tau} \sqrt{\tau^{1-2H} \lambda_H^{-1}}.$$

Let

$$B_t(\theta, x) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) h_s(\theta, x) ds, 0 \leq t \leq \tau,$$

and

$$C_t(\theta, x) = \frac{d}{dw_t^H} \int_\tau^t \kappa_H(t, s) g_s(\theta, x) ds, \tau \leq t \leq T,$$

Let

$$(3.12) \quad V_t(\theta, x) = B_t(\theta, x)I[t \leq \tau] - C_t(\theta, x)I[t > \tau].$$

where $I[A]$ denotes the indicator function of set A . Suppose that the functions B and C are differentiable with respect to θ . Let B' and C' denote the derivatives of B and C with respect to θ . Define

$$(3.13) \quad [\sigma(\theta, \tau)]^2 = \int_0^\tau [B'_t(\theta, x)]^2 dw_t^H + \int_\tau^T [C'_t(\theta, x)]^2 dw_t^H = \int_0^T [A'_t(\theta, \tau, x_0)]^2 dw_t^H.$$

In addition to the conditions (A.1) – (A.5), we assume that the following condition holds:

(A.6) The functions $B_t(\theta, x)$ and $C_t(\theta, x)$ have two continuous bounded derivatives with respect to θ and the first derivatives $B'_t(\theta, X)$ and $C'_t(\theta, X)$ are continuous in X in $L_2([0, T], dw_t^H)$ uniformly in $\theta \in \Theta$ at the point $x = \{x_t, 0 \leq t \leq T\}$. Furthermore for any compact set $K_T = [\tau_1, \tau_2] \subset (t_1, t_2)$,

- (i) $\inf_{\theta \in \Theta, \tau \in K_T} [\int_0^\tau [B_t(\theta, x)]^2 dw_t^H + \int_\tau^T [C_t(\theta, x)]^2 dw_t^H] > 0$,
- (ii) $\inf_{\theta \in \Theta, \tau \in K_T} [V_\tau(\theta, x)]^2 > 0$, and
- (iii) $\inf_{\theta \in \Theta, \tau \in K_T} [\sigma(\theta, \tau)]^2 > 0$.

Let

$$\begin{aligned}
(3.14) \quad L_0(u, v) &= u\xi - \frac{1}{2}u^2[\sigma(\theta, \tau)]^2 + J_{\theta, \tau}^* W_1(v) - \frac{1}{2}J_{\theta, \tau}^{*2}v \text{ if } v \geq 0 \\
&= u\xi - \frac{1}{2}u^2[\sigma(\theta, \tau)]^2 + J_{\theta, \tau}^* W_2(-v) + \frac{1}{2}J_{\theta, \tau}^{*2}v \text{ if } v < 0
\end{aligned}$$

where $\{W_1(v), v \geq 0\}$ and $\{W_2(v), v \geq 0\}$ are independent standard Wiener processes and ξ is a Gaussian random variable with mean zero and variance $[\sigma(\theta, \tau)]^2$.

We now state the main result of this paper.

Theorem 3.1: *Suppose the conditions (A.1) – (A.6) hold. Let θ denote the true drift parameter and τ the true change point. Let $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ denote the maximum likelihood estimator of (θ, τ) based on the observation of the process X satisfying the stochastic differential system defined by (3.1) and (3.2). Then, as $\epsilon \rightarrow 0$, the normalized random vector*

$$(\epsilon^{-1}(\hat{\theta}_\epsilon - \theta), \epsilon^{-2}(\hat{\tau}_\epsilon - \tau))$$

converges in law to a random vector ψ whose distribution is the distribution of location of the maximum of the random field $\{L_0(u, v), -\infty < u, v < \infty\}$ as defined above.

4 Weak convergence of the log-likelihood ratio random field

At first, we state a lemma which gives upper bounds on the difference $X_t - x_t$ and $E_\tau(X_t - x_t)^2$ where the process $\{X_t, 0 \leq t \leq T\}$ satisfies the stochastic differential equation system defined by the equations (3.1) and (3.2) and the function x_t is the solution of the corresponding ordinary differential equation discussed above.

Lemma 4.0: *Let the trend function $S_t(\theta, \tau, x)$ satisfy the conditions (A.1) and (A.2). Then there exist nondecreasing positive functions C_t and $C_{it}, i = 1, 2$ independent of θ, τ such that, with probability one,*

$$(i) \sup_{0 \leq s \leq t} |X_s - x_s| \leq C_t \epsilon \sup_{0 \leq s \leq t} |W_s^H|$$

and

$$(ii) \sup_{0 \leq s \leq t} E_{\theta, \tau}(X_s - x_s)^2 \leq C_{1t} \epsilon^2 E[\sup_{0 \leq s \leq t} |W_s^H|^2] \leq C_{2t} t^{2H} \epsilon^2.$$

Proof: Let $u_t = |X_t - x_t|$. Then by the condition (A.1),

$$\begin{aligned} u_t &\leq \int_0^t |S_v(\theta, \tau, X) - S_v(\theta, \tau, x)| dv + \epsilon |W_t^H| \\ &\leq L \int_0^t \left[\int_0^u u_r dr \right] dv + L \int_0^t u_v dv + \epsilon \sup_{0 \leq s \leq t} |W_s^H|. \end{aligned}$$

Applying the Grownwall' lemma (cf. Lemma 1.11, Kutoyants [16]), it follows that there exists a constant $C_t > 0$ such that

$$\sup_{0 \leq s \leq t} |X_s - x_s| \leq C_t \epsilon \sup_{0 \leq s \leq t} |W_s^H|.$$

In particular, it follows that

$$\begin{aligned} \sup_{0 \leq s \leq t} E_{\theta, \tau} (X_s - x_s)^2 &\leq C_{1t} \epsilon^2 E \left[\sup_{0 \leq s \leq t} |W_s^H|^2 \right] \\ &\leq C_{2t} \epsilon^2 t^{2H} \end{aligned}$$

by Proposition 1.9 in Prakasa Rao [30].

In view of Lemma 4.0 and the condition (A.3), it follows that there exists a constant C_T depending on T but independent of θ, τ , such that

$$(4. 1) \quad \sup_{t_1 \leq \tau, \tau' \leq t_2, \theta \in \Theta} \sup_{0 \leq t \leq T} E_{\theta, \tau} [A_t(\theta, \tau', X) - A_t(\theta, \tau', x)]^2 \leq C_T \epsilon^2.$$

In particular, it follows that

$$(4. 2) \quad \sup_{t_1 \leq \tau \leq t_2, \theta \in \Theta} \sup_{0 \leq t \leq T} E_{\theta, \tau} |\Delta_t - \bar{\Delta}_t|^2 \leq C_T \epsilon^2.$$

Let $P_{\theta, \tau}$ be the probability measure generated by the process Y on the space $C(\Theta \times [0, T])$ associated with the uniform topology when (θ, τ) is the true change point. Consider the log-likelihood ratio random field

$$\begin{aligned} (4. 3) \quad L_\epsilon(u, v) &= \log \frac{dP_{\theta+u\epsilon, \tau+\epsilon^2v}}{dP_{\theta, \tau}} \\ &= \frac{1}{\epsilon} \int_0^T \left[A_t(\theta + u\epsilon, \tau + \epsilon^2v, X) - A_t(\theta, \tau, X) \right] dM_t^H \\ &\quad - \frac{1}{2\epsilon^2} \int_0^T \left[A_t(\theta + u\epsilon, \tau + \epsilon^2v, X) - A_t(\theta, \tau, X) \right]^2 dw_t^H \end{aligned}$$

for fixed u, v such that $0 \leq \tau, \tau + \epsilon^2 v \leq T$ and $\theta, \theta + u\epsilon \in \Theta$.

Let K denote a compact subset of $\Theta \times [0, T]$ such that $(\theta, \tau) \in K$ and $(\theta + \epsilon u, \tau + \epsilon^2 v) \in K$. Let C_K denote the space of continuous functions defined on the set K . Let $K_{\theta \times \tau} = \{(u, v) : (\theta, \tau) \in K \text{ and } (\theta + \epsilon u, \tau + \epsilon^2 v) \in K\}$.

Theorem 4.1: *Under the conditions (A.1) to (A.4), the family of probability measures, generated by the log-likelihood ratio random field $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ associated with the uniform norm topology, converge weakly to the probability measure generated by the random field $\{L_0(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ as $\epsilon \rightarrow 0$.*

From the general theory of weak convergence of probability measures on the space $C_{K_{\theta \times \tau}}$ associated with the uniform norm topology (cf. Billingsley [2], Parthasarathy [25], Prakasa Rao [28]), in order to prove Theorem 4.1, it is sufficient to prove that the finite dimensional distributions of the random field $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ converge to the corresponding finite dimensional distributions of the random field $\{L_0(u, v), (u, v) \in K_{\theta \times \tau}\}$ and the family of probability measures generated by the random fields $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ for different ϵ is tight.

5 Proof of Theorem 4.1

Before we give a proof of Theorem 4.1, we prove some related lemmas.

Lemma 5.1: *Under the conditions (A.1) – (A.4), the finite dimensional distributions of the random field $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ converge to the corresponding finite dimensional distributions of the random field $\{L_0(u, v), (u, v) \in K_{\theta \times \tau}\}$ as $\epsilon \rightarrow 0$.*

Proof: We will first investigate the convergence of the one-dimensional marginal distributions of the random field $L_\epsilon(u, v)$ as $\epsilon \rightarrow 0$.

Suppose $v > 0$. Note that

$$L_\epsilon(u, v) = \frac{1}{\epsilon} \int_0^T \Delta_t dM_t^H - \frac{1}{2\epsilon^2} \int_0^T \Delta_t^2 dw_t^H.$$

and, for $v > 0$,

$$\int_0^T \Delta_t^2 dw_t^H = \int_0^\tau \Delta_t^2 dw_t^H + \int_\tau^{\tau + \epsilon^2 v} \Delta_t^2 dw_t^H + \int_{\tau + \epsilon^2 v}^T \Delta_t^2 dw_t^H.$$

Observe that

$$\begin{aligned}
\int_0^\tau \Delta_t^2 dw_t^H &= \int_0^\tau [B_t(\theta + u\epsilon, X) - B_t(\theta, X)]^2 dw_t^H \\
&= \int_0^\tau [B_t(\theta + u\epsilon, X) - B_t(\theta, X) - u\epsilon B'_t(\theta, X)]^2 dw_t^H \\
&\quad + 2u\epsilon \int_0^\tau (B_t(\theta + u\epsilon, X) - B_t(\theta, X)) B'_t(\theta, X) dw_t^H \\
&\quad - u^2 \epsilon^2 \int_0^\tau [B'_t(\theta, X)]^2 dw_t^H.
\end{aligned}$$

Following arguments in Kutoyants [16], p.50 and p.168 and Lemma 5.1 in Mishra and Prakasa Rao [21], it can be checked that

$$\begin{aligned}
&\frac{1}{\epsilon^2} \int_0^\tau [\Delta_t - u\epsilon B'_t(\theta, X)]^2 dw_t^H \\
&\leq c\epsilon^2 + 2u^2 \int_0^\tau [B'_t(\theta, X) - B'_t(\theta, x)]^2 dw_t^H = o_p(1)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\epsilon^2} \int_{\tau+\epsilon^2v}^T [\Delta_t - u\epsilon C'_t(\theta, X)]^2 dw_t^H \\
&\leq c\epsilon^2 + 2u^2 \int_{\tau+\epsilon^2v}^T [C'_t(\theta, X) - C'_t(\theta, x)]^2 dw_t^H = o_p(1).
\end{aligned}$$

Hence

$$(5.1) \quad \frac{1}{\epsilon^2} \int_0^\tau \Delta_t^2 dw_t^H = u^2 \int_0^\tau [B'_t(\theta, x)]^2 dw_t^H + o_p(1)$$

and

$$(5.2) \quad \frac{1}{\epsilon^2} \int_{\tau+\epsilon^2v}^T \Delta_t^2 dw_t^H = u^2 \int_\tau^T [C'_t(\theta, x)]^2 dw_t^H + o_p(1).$$

Following computations in Lemma 5.1 of Mishra and Prakasa Rao [21], it follows that

$$\begin{aligned}
\frac{1}{\epsilon^2} \int_\tau^{\tau+\epsilon^2v} \Delta_t^2 dw_t^H &= v\lambda_H^{-1} J_{\theta,\tau}^2 \tau^{1-2H} + o_p(1) \\
&= v[J_{\theta,\tau}^*]^2 + o_p(1)
\end{aligned}$$

as $\epsilon \rightarrow 0$. Let us now study the asymptotic behaviour of the random variable

$$\frac{1}{\epsilon} \int_0^T \Delta_t dM_t^H$$

as $\epsilon \rightarrow 0$. Note that

$$\begin{aligned} \frac{1}{\epsilon} \int_0^T \Delta_t dM_t^H &= \frac{1}{\epsilon} \int_0^\tau \Delta_t dM_t^H + \frac{1}{\epsilon} \int_\tau^{\tau+\epsilon^2 v} \Delta_t dM_t^H + \frac{1}{\epsilon} \int_{\tau+\epsilon^2 v}^T \Delta_t dM_t^H \\ &= I_1 + I_2 + I_3 \quad (\text{say}). \end{aligned}$$

Following arguments given Kutoyants [16], p.50 and p.168 and Lemma 5.1 in Mishra and Prakasa Rao [21], it follows that

$$(5.3) \quad I_1 = u \int_0^\tau B'_t(\theta, x) dM_t^H + o_p(1),$$

$$(5.4) \quad I_3 = u \int_\tau^T C'_t(\theta, x) dM_t^H + o_p(1),$$

and I_2 is asymptotically normal with mean zero and variance $[J_{\theta, \tau}^*]^2 v$. Combining the above arguments, it follows that the family of random variables $L_\epsilon(u, v)$ converges in law to the random variable $L_0(u, v)$ defined by (3.14) for any fixed (u, v) as $\epsilon \rightarrow 0$. Similar analysis can be done for the case $v < 0$.

We have proved the convergence of the univariate distributions of the random field $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ as $\epsilon \rightarrow 0$, after proper scaling. Convergence of all the other finite dimensional distributions of the random field $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$, after proper scaling, as $\epsilon \rightarrow 0$, follows by an application of the Cramer-Wold device.

Remarks : In order to prove that a sequence of k -dimensional random vectors \mathbf{X}_n converge in law to a k -dimensional random vector \mathbf{X} as $n \rightarrow \infty$, it is sufficient to prove that the sequence of random variables $\lambda' \mathbf{X}_n$ converges in law to the random variable $\lambda' \mathbf{X}$ for all $\lambda \in R^k$. This is known as the Cramer-Wold technique for converting the problem of the finite dimensional convergence to convergence of one-dimensional random variables. Similar ideas have been applied earlier in proving weak convergence of processes. See Fokianos and Newmann (A goodness-of-fit test for Poisson count processes, *Electronic Journal of Statistics*, Vol.7 (2013), pp. 793-819). We have taken recourse to this technique in the proof given above.

We now state two lemmas which will be used in the following computations. For proofs of these lemmas, see Lemmas 5.2 and 5.3 in Mishra and Prakasa Rao [21].

Lemma 5.2: Let $\{D_t, 0 \leq t \leq T\}$ be a random process such that $\sup_{0 \leq t \leq T} E(D_t^4) \leq \gamma < \infty$.

Then, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$E\left(\left[\int_{\theta_2}^{\theta_1} D_t dt\right]^4\right) \leq |\theta_1 - \theta_2|^3 \int_{\theta_2}^{\theta_1} E[D_t^4] dt \leq \gamma |\theta_1 - \theta_2|^4.$$

The next lemma gives an inequality for the 4-th moment of a stochastic integral with respect to a martingale.

Lemma 5.3: *Let the process $\{f_t, 0 \leq t \leq T\}$ be a random process adapted to a square integrable martingale $\{M_t, \mathcal{F}_t, t \geq 0\}$ with the quadratic variation $\langle M \rangle_t$ such that*

$$\int_0^T E(f_s^4) d\langle M \rangle_s < \infty.$$

Then

$$E\left(\left(\int_0^T f_t dM_t\right)^4\right) \leq 36 \langle M \rangle_T \int_0^T E(f_t^4) d\langle M \rangle_t.$$

and, in general, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$E\left(\left(\int_{\theta_2}^{\theta_1} f_t dM_t\right)^4\right) \leq 36(\langle M \rangle_{\theta_1} - \langle M \rangle_{\theta_2}) \int_{\theta_2}^{\theta_1} E[f_t^4] d\langle M \rangle_t.$$

Lemma 5.4: *Let $\Gamma_\epsilon(u, v) = \exp\{L_\epsilon(u, v)\}$. Suppose the conditions (A.1) – (A.6) hold. Then, for any $R > 0$, there exist a constant $C > 0$ such that*

$$E_{\theta, \tau} \left| \Gamma_\epsilon^{\frac{1}{4}}(u_2, v_2) - \Gamma_\epsilon^{\frac{1}{4}}(u_1, v_1) \right|^4 \leq C[(u_1 - u_2)^4 + (v_1 - v_2)^2], |u_i|, |v_i| \leq R, i = 1, 2.$$

Proof : Without loss of generality, let $v_1 > v_2$,

$$\delta_t = A_t(\theta + \epsilon u_1, \tau + \epsilon^2 v_1, X) - A_t(\theta + \epsilon u_2, \tau + \epsilon^2 v_2, X)$$

and

$$\bar{\delta}_t = A_t(\theta + \epsilon u_1, \tau + \epsilon^2 v_1, x) - A_t(\theta + \epsilon u_2, \tau + \epsilon^2 v_2, x).$$

Recall the notation $\tau + \epsilon^2 v_1 = \zeta_1, \tau + \epsilon^2 v_2 = \zeta_2$ and $\theta + \epsilon u_1 = \beta_1, \theta + \epsilon u_2 = \beta_2$ used earlier.

Let

$$R_t = \exp\left[\frac{1}{4\epsilon} \int_0^t \delta_s dM_s^H - \frac{1}{8\epsilon^2} \int_0^t \delta_s^2 dw_s^H\right], R_0 = 1.$$

Note that the process R_t is the process $\left(\frac{dP_{(\beta_1, \zeta_1)}}{dP_{(\beta_2, \zeta_2)}}(X)\right)^{\frac{1}{4}}$ and, by the Ito formula, we have

$$dR_t = -\frac{3}{(32)\epsilon^2}\delta_t^2 R_t dw_t^H + \frac{1}{4\epsilon}\delta_t R_t dM_t^H.$$

Hence

$$R_T = 1 - \frac{3}{(32)\epsilon^2}\int_0^T \delta_t^2 R_t dw_t^H + \frac{1}{4\epsilon}\int_0^T \delta_t R_t dM_t^H.$$

Note that

$$\begin{aligned} & E_{\theta, \tau} \left| \Gamma_{\epsilon}^{\frac{1}{4}}(u_2, v_2) - \Gamma_{\epsilon}^{\frac{1}{4}}(u_1, v_1) \right|^4 \\ &= E_{\theta, \tau} \left(\frac{dP_{(\beta_2, \zeta_2)}}{dP_{\theta, \tau}} |1 - R_T|^4 \right) = E_{(\beta_2, \zeta_2)} (|1 - R_T|^4) \\ &\leq C \frac{1}{\epsilon^8} E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t^2 R_t dw_t^H \right|^4 + C \frac{1}{\epsilon^4} E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t R_t dM_t^H \right|^4 \end{aligned}$$

where C is an absolute constant. In order to get the bounds for the expectations of the integrals in the above inequality, we now use the Lemmas 5.2 and 5.3.

Let us now estimate the term

$$E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t^2 R_t dw_t^H \right|^4.$$

Suppose that $v_1 > v_2$. Let $v = v_1 - v_2 > 0$. Then

$$\begin{aligned} & E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t^2 R_t dw_t^H \right|^4 \\ &\leq c E_{\beta_2, \zeta_2} \left| \int_0^{\tau} \delta_t^2 R_t dw_t^H + \int_{\tau+v\epsilon^2}^T \delta_t^2 R_t dw_t^H \right|^4 \\ &\quad + c E_{\beta_2, \zeta_2} \left| \int_{\tau}^{\tau+v\epsilon^2} \delta_t^2 R_t dw_t^H \right|^4 \end{aligned}$$

for some absolute constant $c > 0$. Note that

$$I_1 \equiv E_{\beta_2, \zeta_2} \left| \int_{\tau}^{\tau+v\epsilon^2} \delta_t^2 R_t dw_t^H \right|^4$$

$$\begin{aligned}
&= E_{\beta_2, \zeta_2} \left| \int_{\tau}^{\tau+v\epsilon^2} \delta_t^2 R_t \lambda_H^{-1} (2-2H) t^{1-2H} dt \right|^4 \\
&\leq c v^3 \epsilon^6 \int_{\tau}^{\tau+v\epsilon^2} E_{\beta_2, \zeta_2} |\delta_t^2 R_t|^4 t^{4-8H} dt
\end{aligned}$$

by Lemma 5.2. Now

$$\sup_{0 \leq t \leq T} E_{\beta_2, \zeta_2} ([\delta_t^2 R_t]^4) = \sup_{0 \leq t \leq T} E_{\beta_1, \zeta_1} (\delta_t^8) < \infty$$

by the condition (A.4)(i) since

$$R_t = \left(\frac{dP_{(\beta_1, \zeta_1)}}{dP_{(\beta_2, \zeta_2)}}(X) \right)^{1/4}.$$

As a consequence of the Lemma 5.2 and the upper bound obtained above, it follows that

$$\begin{aligned}
I_1 &= E_{\beta_2, \zeta_2} \left| \int_{\tau}^{\tau+v\epsilon^2} \delta_t^2 R_t dw_t^H \right|^4 \\
&\leq c v^3 \epsilon^6 |\tau^{5-8H} - (\tau + v\epsilon^2)^{5-8H}| \\
&\leq c \epsilon^6 (v_1 - v_2)^3 \tau^{4-8H} \epsilon^2 (v_1 - v_2) \\
&\leq c (v_1 - v_2)^4 \epsilon^8
\end{aligned}$$

for some constant $c > 0$ depending on H, T and Θ . Let us now estimate

$$I_2 = E_{\beta_2, \zeta_2} \left| \int_0^{\tau} \delta_t^2 R_t dw_t^H \right|^4.$$

Note that

$$\begin{aligned}
I_2 &\equiv E_{\beta_2, \zeta_2} \left| \int_0^{\tau} \delta_t^2 R_t dw_t^H \right|^4 \\
&= E_{\beta_2, \zeta_2} \left| \int_0^{\tau} \delta_t^2 R_t \lambda_H^{-1} (2-2H) t^{1-2H} dt \right|^4 \\
&\leq c \tau^3 \int_0^{\tau} E_{\beta_2, \zeta_2} [\delta_t^8 R_t^4] t^{4-8H} dt \\
&= c \tau^3 \int_0^{\tau} E_{\beta_1, \zeta_1} [\delta_t^8] t^{4-8H} dt \\
&\leq c \tau^{8-8H} \sup_{\theta, \tau, t} E_{\theta, \tau} [\delta_t^8] \\
&\leq c \tau^{8-8H} \epsilon^8 (u_2 - u_1)^8 \\
&\leq c \epsilon^8 (u_2 - u_1)^8
\end{aligned}$$

since

$$\delta_t = B_t(\theta + \epsilon u_1, x) - B_t(\theta + \epsilon u_2, x)$$

on the interval $[0, \tau]$ and the condition (A.6) holds. Furthermore,

$$\begin{aligned} I_3 &\equiv E_{\beta_2, \zeta_2} \left| \int_{\tau + \epsilon^2 v}^T \delta_t^2 R_t d w_t^H \right|^4 \\ &= E_{\beta_2, \zeta_2} \left| \int_{\tau + \epsilon^2 v}^T \delta_t^2 R_t \lambda_H^{-1} (2 - 2H) t^{1-2H} dt \right|^4 \\ &\leq c \int_{\tau + \epsilon^2 v}^T E_{\beta_2, \zeta_2} [\delta_t^8 R_t^4] t^{4-8H} dt \\ &= c \int_{\tau + \epsilon^2 v}^T E_{\beta_1, \zeta_1} [\delta_t^8] t^{4-8H} dt \\ &\leq c \sup_{\theta, \tau, t} E_{\theta, \tau} [\delta_t^8] \\ &\leq c \epsilon^8 (u_2 - u_1)^8 \end{aligned}$$

by (A.6) since

$$\delta_t = C_t(\theta + \epsilon u_1, x) - C_t(\theta + \epsilon u_2, x)$$

on the interval $[\tau + \epsilon^2 v, T]$ and the condition (A.6) holds

Let us now consider estimation of the term

$$E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t R_t dM_t^H \right|^4.$$

Note that

$$\begin{aligned} E_{\beta_2, \zeta_2} \left| \int_0^T \delta_t R_t dM_t^H \right|^4 &\leq c E_{\beta_2, \zeta_2} \left(\left| \int_0^\tau \delta_t R_t dM_t^H + \int_{\tau + \epsilon^2 v}^T \delta_t R_t dM_t^H \right|^4 \right) \\ &\quad + c E_{\beta_2, \zeta_2} \left(\left| \int_\tau^{\tau + \epsilon^2 v} \delta_t R_t dM_t^H \right|^4 \right). \end{aligned}$$

Let

$$I'_1 \equiv E_{\beta_2, \zeta_2} \left| \int_\tau^{\tau + v \epsilon^2} \delta_t^2 R_t dM_t^H \right|^4.$$

Then

$$I'_1 \equiv E_{\beta_2, \zeta_2} \left| \int_\tau^{\tau + v \epsilon^2} \delta_t^2 R_t dM_t^H \right|^4$$

$$\begin{aligned}
&\leq c(w_{\tau+\epsilon^2 v}^H - w_\tau^H) \int_\tau^{\tau+v\epsilon^2} E_{\beta_2, \zeta_2} [\delta_t R_t]^4 dw_t^H \\
&\leq c(w_{\tau+\epsilon^2 v}^H - w_\tau^H) \int_\tau^{\tau+v\epsilon^2} E_{\beta_2, \zeta_2} [\delta_t R_t]^4 \lambda_H^{-1} (2-2H) t^{1-2H} dt \\
&\leq c[(\tau + \epsilon^2 v)^{2-2H} - \tau^{2-2H}] \int_\tau^{\tau+v\epsilon^2} E_{\beta_1, \zeta_1} [\delta_t]^4 t^{1-2H} dt \\
&\leq c[(\tau + \epsilon^2 v)^{2-2H} - \tau^{2-2H}]^2 \\
&\leq c(v_1 - v_2)^2 \epsilon^4
\end{aligned}$$

by Lemma 5.3 in view of the condition (A.4) . Similarly

$$\begin{aligned}
I'_2 &\equiv E_{\beta_2, \zeta_2} \left| \int_0^\tau \delta_t^2 R_t dM_t^H \right|^4 \\
&\leq c w_\tau^H \int_0^\tau E_{\beta_2, \zeta_2} [\delta_t R_t]^4 dw_t^H \\
&\leq c w_\tau^H \int_0^\tau E_{\beta_2, \zeta_2} [\delta_t R_t]^4 \lambda_H^{-1} (2-2H) t^{1-2H} dt \\
&\leq c \tau^{2-2H} \int_0^\tau E_{\beta_1, \zeta_1} [\delta_t]^4 t^{1-2H} dt \\
&\leq c(u_1 - u_2)^4 \epsilon^4
\end{aligned}$$

by Lemma 5.3 in view of the condition (A.6) and

$$\begin{aligned}
I'_3 &\equiv E_{\beta_2, \zeta_2} \left| \int_{\tau+\epsilon^2 v}^T \delta_t^2 R_t dM_t^H \right|^4 \\
&\leq c(w_T^H - w_{\tau+\epsilon^2 v}^H) \int_{\tau+\epsilon^2 v}^T E_{\beta_2, \zeta_2} [\delta_t R_t]^4 dw_t^H \\
&\leq c(w_T^H - w_{\tau+\epsilon^2 v}^H) \int_{\tau+\epsilon^2 v}^T E_{\beta_2, \zeta_2} [\delta_t R_t]^4 \lambda_H^{-1} (2-2H) t^{1-2H} dt \\
&\leq c(T^{2-2H} - (\tau + \epsilon^2 v)^{2-2H}) \int_{\tau+\epsilon^2 v}^T E_{\beta_1, \zeta_1} [\delta_t]^4 t^{1-2H} dt \\
&\leq c(u_1 - u_2)^4 \epsilon^4.
\end{aligned}$$

by Lemma 5.3 in view of the condition (A.6). Combining the above estimates, we obtain that

$$\sup_{|u_i| \leq R, |v_i| \leq R} [(u_1 - u_2)^4 + (v_1 - v_2)^2]^{-1} E_{\theta, \tau} |\Gamma_\epsilon^{1/4}(u_2, v_2) - \Gamma_\epsilon^{1/4}(u_1, v_1)|^4 < c < \infty$$

which proves the tightness from results in Prakasa Rao [28] or Neuhaus [22].

As a consequence of the Lemma 5.4, it follows that the family of probability measures generated by the processes $\{\Gamma_\epsilon^{\frac{1}{4}}(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ with uniform topology is tight from the results in Billingsley [2] (cf. Prakasa Rao [27, 28]) and hence the family of probability measures generated by the processes $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ is tight.

Lemmas 5.1 and 5.4 together imply that that the family of probability measures generated by the processes $\{L_\epsilon(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ converge weakly to the probability measure generated by the processes $\{L_0(u, v), (u, v) \in K_{\theta \times \tau}\}$ on $C_{K_{\theta \times \tau}}$ from the general theory of weak convergence of probability measures on complete separable metric spaces(cf. Billingsley [2], Parthasarathy [25], Prakasa Rao [27] and Ibragimov and Khasminskii [12]). This completes the proof of Theorem 4.1.

It remains to show that the maximum likelihood estimator $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ will lie in a compact set K with probability tending to one as $\epsilon \rightarrow 0$.

The following maximal inequality is proved in Lemma 5.6 in Mishra and Prakasa Rao [21] using the Slepian's lemma (cf. Leadbetter et al. [18] and Matsui and Shieh [20]). We will use it in the sequel.

Lemma 5.5: *For any $\lambda > 0$,*

$$E[\exp\{\lambda \max_{0 \leq t \leq T} |W_t^H|\}] \leq 1 + \lambda \sqrt{2\pi T^{2H}} \exp\left\{\frac{\lambda^2 T^{2H}}{2}\right\}.$$

We now apply Lemma 5.5 to get the following result.

Lemma 5.6: *Suppose the conditions (A.1) to (A.6) hold. Let $\Gamma_\epsilon(u, v) = \exp\{L_\epsilon(u, v)\}$, $u, v \in R$. Then, for any compact set $K \subset [0, T] \times \Theta$, and for any $0 < p < 1$, there exists a positive constant C such that*

$$(5. 5) \quad \sup_{(\theta, \tau) \in K} E_{\theta, \tau}[(\Gamma_\epsilon(u, v))^p] \leq e^{-C g(u, v)}$$

for all u, v where $g(u, v) = k_1|u|^\alpha + k_2|v|^\beta$ for some $k_1 > 0$ and $k_2 > 0$.

Proof: Now, for any $0 < p < 1$, we will now estimate $E_{\theta, \tau}(\Gamma_\epsilon(u, v))^p$. For convenience, let $u \in R$ and $v > 0$ and let

$$F_1 \equiv \int_0^T \Delta_t dM_t^H$$

and

$$F_2 \equiv \int_0^T \Delta_t^2 dw_t^H.$$

Let q be such that $p^2 < q < p$. Then

$$\begin{aligned} E_{\theta,\tau}[(\Gamma_\epsilon(u,v))^p] &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{p}{2\epsilon^2}F_2\}] \\ &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2 - \frac{(p-q)}{2\epsilon^2}F_2\}]. \end{aligned}$$

Let

$$G_1 = \exp\{-\frac{(p-q)}{2\epsilon^2}F_2\}$$

and

$$G_2 = \exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2\}.$$

Then

$$\begin{aligned} E_{\theta,\tau}[(\Gamma_\epsilon(u,v))^p] &= E_{\theta,\tau}[G_1 G_2] \\ &\leq (E_{\theta,\tau}[G_1^{p_1}])^{1/p_1} (E_{\theta,\tau}[G_2^{p_2}])^{1/p_2} \end{aligned}$$

by the Holder inequality for any p_1 and p_2 such that $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Choose $p_2 = \frac{q}{p^2} > 1$. Then $p_1 = \frac{q}{q-p^2}$. Observe that

$$\begin{aligned} E_{\theta,\tau}[G_2^{p_2}] &= E_{\theta,\tau}[\exp\{p_2(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \\ &= E_{\theta,\tau}[\exp\{\frac{q}{p^2}(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \\ &= E_{\theta,\tau}[\exp\{\frac{1}{\epsilon}\frac{q}{p}F_1 - \frac{1}{2\epsilon^2}\frac{q^2}{p^2}F_2\}]. \end{aligned}$$

The random variable, under the expectation sign in the last line, is the Radon-Nikodym derivative of two probability measures which are absolutely continuous with respect to each other by the Girsanov's theorem for martingales. Hence the expectation is equal to one. Hence

$$\begin{aligned} E_{\theta,\tau}[(\Gamma_\epsilon(u,v))^p] &\leq (E[\exp\{-\frac{p_1(p-q)}{2\epsilon^2}F_2\}])^{1/p_1} \\ &= (E[\exp\{-\gamma\epsilon^{-2}F_2\}])^{1/p_1}. \end{aligned}$$

where $\gamma = \frac{q(p-q)}{2(q-p^2)} > 0$. Let us now estimate $E_{\theta,\tau}[e^{-\gamma\epsilon^{-2}F_2}]$. Applying the inequality

$$a^2 \geq b^2 - 2|b(a-b)|,$$

it follows that

$$\begin{aligned} E_{\theta,\tau}[e^{-\gamma\epsilon^{-2}F_2}] &\leq \exp\{-\gamma\epsilon^{-2} \int_0^T \bar{\Delta}_t^2 dw_t^H\} \times \\ &\quad \times E_{\theta,\tau}[\exp\{2\gamma\epsilon^{-2}(\int_0^T (A_t(\theta + \epsilon u, \tau + \epsilon^2 v, X) - A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x))| \\ &\quad + |A_t(\theta, \tau, X) - A_t(\theta, \tau, x)|)|A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)|dw_t^H\}]. \end{aligned}$$

We now get an upper bound on the term under the expectation sign on the right side of the above inequality. Observe that there exists a constant $L > 0$, such that,

$$\begin{aligned} &\int_0^T [A_t(\theta, \tau, X) - A_t(\theta, \tau, x)]^2 dw_t^H \\ &\leq L^2 [\int_0^T \sup_{0 \leq s \leq t} |X_s - x_s|^2 dw_t^H] \\ &\leq L^2 \sup_{0 \leq t \leq T} |X_t - x_t|^2 [\int_0^T dw_t^H] \\ &\leq L^2 \epsilon^2 e^{2LT} [\int_0^T dw_t^H] \sup_{0 \leq t \leq T} |W_t^H|^2 \quad (\text{by Lemma 4.0}) \\ &\leq C \epsilon^2 L^2 e^{2LT} T^{2-2H} \sup_{0 \leq t \leq T} |W_t^H|^2. \end{aligned}$$

for some constant $C > 0$ possibly depending on T, H and Θ . Therefore, under the condition (A.5), for any $\tau' \in [t_1, t_2]$, and $\theta \in \Theta$, an application of the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \sup_{t_1 \leq \tau, \tau' \leq t_2, \theta, \theta' \in \Theta} &[\int_0^T |A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)| |A_t(\theta', \tau', X) - A_t(\theta', \tau', x)| dw_t^H]^2 \\ &\leq CL^2 \epsilon^4 g(u, v) e^{2LT} T^{2-2H} \sup_{0 \leq t \leq T} |W_t^H|^2. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{t_1 \leq \tau, \tau' \leq t_2, \theta, \theta' \in \Theta} &[\int_0^T |A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)| |A_t(\theta', \tau', X) - A_t(\theta', \tau', x)| dw_t^H] \\ &\leq C \epsilon^2 [g(u, v)]^{1/2} \sup_{0 \leq t \leq T} |W_t^H|. \end{aligned}$$

Therefore

$$\begin{aligned}
& \sup_{t_1 \leq \tau \leq t_2, \theta \in \Theta} E_{\theta, \tau} [\exp\{2\gamma\epsilon^{-2}(\int_0^T (|A_t(\theta + \epsilon u, \tau + \epsilon^2 v, X) - A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x)| + \\
& \quad + |A_t(\theta, \tau, X) - A_t(\theta, \tau, x)|) |A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)| dw_t^H)\}] \\
& \leq E_{\theta, \tau} [\exp\{C\gamma[g(u, v)]^{1/2} \sup_{0 \leq t \leq T} |W_t^H|\}] \\
& \leq 1 + \gamma C[g(u, v)]^{1/2} \sqrt{2\pi T^{2H}} \exp\left\{\frac{c\gamma^2 T^{2H} g(u, v)}{2}\right\}
\end{aligned}$$

by Lemma 5.5. Applying arguments similar to those in Lemma 2.4 in Kutoyants [16], we get that

$$\sup_{(\theta, \tau) \in K} E_{\theta, \tau} [\Gamma_\epsilon^p(u, v)] \leq e^{-C g(u, v)}$$

for some positive constant $C > 0$ depending on T, H and Θ .

An application of Lemma 5.5, proved earlier, shows that the maximum likelihood estimator $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ will lie in the compact set K with probability tending to one as $\epsilon \rightarrow 0$ from Theorem 5.1 in Chapter 1, p.42 of Ibragimov and Khasminskii [12] (cf. Kutoyants [15]).

We now give a proof of Theorem 3.1.

Proof of Theorem 3.1: Let C_K denote the family of continuous functions defined on a compact set K in R^2 . In view of Theorem 4.1, it follows that the family of probability measures generated by the random fields $\{L_\epsilon(u, v), (u, v) \in K\}, \epsilon > 0$ on C_K converge weakly to the probability measure generated by the random field $\{L_0(u, v), (u, v) \in K\}$ on C_K as $\epsilon \rightarrow 0$. Let $(\hat{u}_\epsilon, \hat{v}_\epsilon)$ denote the infimum of the points of the maxima of the random field $\{L_\epsilon(u, v), (u, v) \in K\}, \epsilon > 0$ on C_K . Let (u_0, v_0) denote the location of the maxima of the process $\{L_0(u, v), (u, v) \in K\}$ on C_K . The location (u_0, v_0) of the maxima is unique almost surely by the property of Gaussian random fields. Since the random field $\{L_\epsilon(u, v), (u, v) \in K\}, \epsilon > 0$ on C_K converge weakly to the random field $\{L_0(u, v), (u, v) \in K\}$ on C_K as $\epsilon \rightarrow 0$, by the continuous mapping theorem, it follows that the distribution of $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ appropriately normalized converges in law to the distribution of (u_0, v_0) by the continuous mapping theorem (cf. Billingsley [2]). Lemma 5.6 implies that the random variable $(\hat{u}_\epsilon, \hat{v}_\epsilon) = (\epsilon^{-1}(\hat{\theta}_\epsilon - \theta), \epsilon^{-2}(\hat{\tau}_\epsilon - \tau)) \in K$ with probability tending to one as $\epsilon \rightarrow 0$. Applying arguments similar to those in Theorem 10.1 in Chapter II, p.103 of Ibragimov and Khasminskii [12] (cf. Prakasa Rao [26]), we obtain the following result. Let (θ, τ) be the true change

point. As a consequence of the arguments and the discussion given above, it follows that the random variable

$$(\hat{u}_\epsilon, \hat{v}_\epsilon) = (\epsilon^{-1}(\hat{\theta}_\epsilon - \theta), \epsilon^{-2}(\hat{\tau}_\epsilon - \tau))$$

converges in law to the distribution of the random variable (u_0, v_0) , the location of the maximum of the random field $\{L_0(u, v), -\infty < u, v < \infty\}$, as $\epsilon \rightarrow 0$.

Remarks : We have assumed that the Hurst index H of the driving force for the fractional diffusion process is known through out the earlier discussions. It would be interesting to find out the asymptotic behaviour of a suitably transformed maximum likelihood estimator $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ when H is unknown by using an estimator \hat{H}_n of H and studying the asymptotic behaviour of corresponding plug-in-estimator as $\epsilon \rightarrow 0$.

Example : Suppose the process $\{X_t, 0 \leq t \leq T\}$ satisfies the stochastic differential system

$$\begin{aligned} dX(t) &= g\theta dt + \epsilon dW_t^H, 0 \leq t \leq \tau \\ dX(t) &= h\theta dt + \epsilon dW_t^H, \tau < t \leq T \end{aligned}$$

where g and h are known constants with $g \neq h$ and $\theta \neq 0$. This is the problem of estimation of the change point for a fractional Brownian motion with a linear shift and a change in shift. It can be seen that the drift functions $S_t(\theta, \tau, x)$ and $A_t(\theta, \tau, x)$ do not depend on x in this example and the conditions (A.1) – (A.3) and (A.4)(i) hold. Note that, in this example, we can directly check that

$$\frac{1}{\epsilon^2} \int_\tau^{\tau+\epsilon^2 v} \bar{\Delta}_t^2 dw_t^H = v \lambda_H^{-1} J_{\theta, \tau}^2 \tau^{1-2H} + o(1)$$

where

$$J_{\theta, \tau}^2 = (g - h)^2 \theta^2$$

and

$$(5.6) \quad J_{\theta, \tau}^* = J_{\theta, \tau} \sqrt{\tau^{1-2H} \lambda_H^{-1}}.$$

This can be seen by checking that

$$\frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) ds = 1$$

from results in Kleptsyna and Le Breton [13]. We now verify condition (A.5). Observe that

$$\begin{aligned}
& \epsilon^{-2} \int_0^T [A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)]^2 dw_t^H \\
&= \epsilon^{-2} \int_0^\tau [A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)]^2 dw_t^H \\
&\quad + \epsilon^{-2} \int_\tau^{\tau + \epsilon^2 v} [A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)]^2 dw_t^H \\
&\quad + \epsilon^{-2} \int_{\tau + \epsilon^2 v}^T [A_t(\theta + \epsilon u, \tau + \epsilon^2 v, x) - A_t(\theta, \tau, x)]^2 dw_t^H \\
&= R_1 + R_2 + R_3 \text{ (say)}.
\end{aligned}$$

Following the computations made earlier, it can be checked that

$$R_1 = g^2 u^2 \lambda_H^{-1} \tau^{2-2H} + o(1),$$

$$R_2 = v J_{\theta, \tau}^{*2} + o(1),$$

and

$$R_3 = h^2 u^2 \lambda_H^{-1} [T^{2-2H} - \tau^{2-2H}] + o(1)$$

as $\epsilon \rightarrow 0$. It can be checked that the condition (A.5) holds with $g(u, v) = c_1 |u|^2 + c_2 |v|$ for some positive constants $c_1 > 0$ and $c_2 > 0$. Note that $B_t(\theta, x) = g\theta, 0 \leq t \leq \tau$ and

$$C_t(\theta, x) = h\theta \frac{d}{dw_t^H} \int_\tau^t \kappa_H(t, s) ds, \tau < t \leq T.$$

It is easy to see that the condition (A.6) holds for these functions $B_t(\theta, x)$ and $C_t(\theta, x)$. Let $(\hat{\theta}_\epsilon, \hat{\tau}_\epsilon)$ be the maximum likelihood estimator. An application of Theorem 3.1 implies that the random variable $(\epsilon^{-1}(\hat{\theta}_\epsilon - \theta), \epsilon^{-2}(\hat{\tau}_\epsilon - \tau))$ has a limiting distribution as $\epsilon \rightarrow 0$.

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