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Author (s): **B. L. S. Prakasa Rao**

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Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org

ASSOCIATED SEQUENCES AND DEMIMARTINGALES

B.L.S. Prakasa Rao

CR RAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)
University of Hyderabad Campus, HYDERABAD, INDIA 500046

E-mail:blsprao@gmail.com

Associated Sequences

- The concept of association of random variables was introduced by Esary, Proschan and Walkup (1967). In several situations, for example, in reliability and survival analysis, the random variables of lifetimes involved are not independent but are generally associated. In the classical case of statistical inference, the observed random variables of interest are generally assumed to be independent and identically distributed. However in several real life situations, the random variables need not be independent.

Associated Sequences

- In reliability studies, there are structures in which the components share the load, so that failure of one component results in increased load on each of the remaining components. Failure of one component will adversely effect the performance of all the minimal path structures containing it. In such a model, the random variables of interest are not independent but are 'associated'. We give a short review of probabilistic properties of associated sequences of random variables. A detailed study of such random variables is given in Prakasa Rao " Associated Sequences, Demimartingales and Nonparametric Inference", Birkhauser, Springer, Basel (2012).

Associated Sequences

- Hoeffding (1940) proved the following result.

  Theorem : (Hoeffding identity) Let (X, Y) be a bivariate random vector such that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Then

$$Cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) dx dy \qquad (1)$$

  where,

$$\begin{aligned} H(x, y) &= P[X > x, Y > y] - P[X > x]P[Y > y] \\ &= P[X \leq x, Y \leq y] - P[X \leq x]P[Y \leq y]. \qquad (2) \end{aligned}$$

Associated Sequences

- **Proof** : Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random variables. Then,

$$2[E(X_1 Y_1) - E(X_1)E(Y_1)]$$
$$= E[(X_1 - X_2)(Y_1 - Y_2)]$$
$$= E \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [I(u, X_1) - I(u, X_2)][I(v, Y_1) - I(v, Y_2)]dudv$$

where $I(u, a) = 1$ if $u \leq a$ and 0 otherwise. The result follows by taking the expectation under the integral sign.

Associated Sequences

- A generalized Hoeffding identity has been proved for multidimensional random vectors by Block and Fang (1983). Newman (1980) showed that for any two functions $h(\cdot)$ and $g(\cdot)$ with $E[h(X)]^2 < \infty$ and $E[g(Y)]^2 < \infty$ and finite derivatives $h'(\cdot)$ and $g'(\cdot)$,

$$\text{Cov}(h(X), g(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h'(x)g'(y)H(x,y)dx\,dy \qquad (3)$$

  Yu (1993) extended the above relation to even dimensional random vectors. Prakasa Rao (1998) further extended this identity following Queseda-Molina (1992).

Associated Sequences

- As a departure from independence, a bivariate notion of positive quadrant dependence was introduced by Lehmann (1966).

  Definition : A pair of random variables (X, Y) is said to be positively quadrant dependent (PQD) if

  $$P[X \leq x, Y \leq y] \geq P[X \leq x]P[Y \leq y] \text{ for all } x, y \tag{4}$$

  or equivalently

  $$H(x, y) \geq 0, \quad x, y \in R. \tag{5}$$

Associated Sequences

- It can be shown that the inequality given in (5) is equivalent to the following:
  for any pair of non-decreasing functions $h(.)$ and $g(.)$ on R

  $$Cov(h(X), g(Y)) \geq 0.$$

  A stronger condition is that, for a pair of random variables (X, Y) and for any two real coordinate-wise non-decreasing functions $h(.,.)$ and $g(.,.)$ on R^2,

  $$Cov(h(X, Y), g(X, Y)) \geq 0.$$

Associated Sequences

- As a natural multivariate extension of (1.7), the following concept of association was introduced by Esary, Proschan and Walkup (1967).

  Definition : A collection of random variables $\{X_n, n \geq 1\}$ is said to be *associated* if for every n and for every choice of coordinate-wise non-decreasing functions h and g from R^n to R,

  $$Cov(h(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$$

  whenever it exists.

Examples of Associated Sequences

- It is easy to see that any set of independent random variables with finite variance is associated (cf. Esary, Proschan and Walkup (1967)). Associated random variables arise in reliability, statistical mechanics, percolation theory etcetra.

  (i) Let $\{X_i', i \geq 1\}$ be independent and identically distributed (i.i.d.) random variables and Y be independent of $\{X_i', i \geq 1\}$. Then $\{X_i = X_i' + Y, i \geq 1\}$ are associated. Thus, independent random variables subject to the same stress are associated (cf. Barlow and Proschan (1975)).

  For an application to modeling dependent competing risks, see Bagai and Prakasa Rao (1992).

Examples of Associated Sequences

- (ii) Order statistics corresponding to a finite set of independent random variables are associated.

  (iii) Positively correlated normal random variables are associated (cf. Pitt (1982)).

Examples of Associated Sequences

- (iv) Suppose a random vector $(X_1, ..., X_m)$ has a multivariate exponential distribution $F(x_1, \ldots, x_m)$ (cf. Marshall and Olkin (1967)). Then the components $X_1, ..., X_m$ are associated.

Examples of Associated Sequences

- (v) Let $\{X_1, \ldots, X_n\}$ be jointly α-stable random variables, $0 < \alpha < 2$. Then Lee, Rachev and Samorodnitsky (1990) discussed necessary and sufficient conditions under which $\{X_1, \ldots, X_n\}$ are associated.

  (vi) Let $\{e_k : k = ..., -1, 0, 1, ...\}$ be a sequence of independent random variables with zero mean and unit variance. Let $\{w_j : j = 0, 1, ...\}$ be a sequence of non-negative real numbers such that $\sum_{j=0}^{\infty} w_j < \infty$. Define $X_k = \sum_{j=0}^{\infty} w_j e_{k-j}$. Then the sequence $\{X_k\}$ is associated. (Nagaraj and Reddy (1993)).

Examples of Associated Sequences

- (vii) Let $\{X_k\}$ be a stationary autoregressive process of order p given by $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + e_t$ where $\{e_t\}$ is a sequence of independent random variables with zero mean and unit variance.Then $\{X_k\}$ is associated if $\phi_i \geq 0, 1 \leq i \leq p$. Suppose $p = 1$ and $\phi_1 < 0$. Then $\{X_{2^k}\}$ and $\{X_{2^k+1}\}$ are associated sequences (Nagaraj and Reddy (1993)).

Examples of Associated Sequences

- (viii) Consider the following network. Suppose customers arrive according to a Poisson process with rate λ and all the customers enter the node 1 initially. Further suppose that the service times at the nodes are mutually independent and exponentially distributed and customers choose either the route $r_1 : 1 \to 2 \to 3$ or $r_2 : 1 \to 3$ according to a Bernoulli process with probability p of choosing r_1. The arrival and the service processes are mutually independent. Let S_1 and S_3 be the sojourn times at the nodes 1 and 3 of a customer that follows the route r_1. Foley and Kiessler (1989) showed that S_1 and S_3 are associated.

Examples of Associated Sequences

- (ix) Let $\{X_n\}$ be a discrete time homogeneous Markov chain. Then $\{X_n\}$ is said to be a monotone Markov chain if $Pr[X_{n+1} \geq y | X_n = x]$ is non-decreasing in x for each fixed y. Daley (1968) showed that a monotone Markov chain is associated.

  (x) Consider a system of k components $1, \ldots, k$, all new at time 0 with life-lengths $T_1, \ldots, T_k$. Arjas and Norros (1984) discussed a set of conditions under which the life-lengths are associated.

Examples of Associated Sequences

- (xi) Consider a system of N non-renewable components in parallel. Let T_i denote the life-time of a component $i, \ i = 1, \ldots, N$. Suppose the environment is represented by a real valued stochastic process $Y = \{Y_t, t \geq 0\}$ which is external to the failure mechanism. Assume that given Y, the life-times T_i are independent and let

$$\lim_{\tau \to 0} \frac{1}{\tau} \Pr[t \leq T_i \leq t + \tau | T_i > t, Y] = \eta_i(t, Y_t), \ i = 1, \ldots, N$$

where each $\eta_i(t, y)$ is a positive continuous function of $t > 0$ and real y. Assume that $\eta_i(t, y)$ are all increasing (or decreasing) in y. Further, let

$$Q_i(t_i) = \int_0^{t_i} \eta_i(u, Y_u) du, \ t_i \geq 0, \ i = 1, \ldots, N.$$

Examples of Associated Sequences

- Thus $Q_i(t_i)$ is the total risk incurred by the component i from the starting time to time t_i. Lefevre and Milhaud (1990) showed that if Y is associated , then the life-lengths $T_1, \ldots, T_n$, are associated as well as the random variables $Q_1(t_1), \ldots, Q_N(t_N)$ are associated.

Examples of Associated Sequences

- Pitt (1982) proved the following result for the components of a mutivariate normal distribution with mean zero and covariance matrix Σ.
  Theorem : Let $\underline{X} = (X_1, \ldots, X_k)$ be $N_k(0, \Sigma),\ \Sigma = ((\sigma_{ij}))$ where Cov $(X_i, X_j) = \sigma_{ij}$. Then a necessary and sufficient condition for $(X_1, \ldots, X_k)$ to be associated is that $\sigma_{ij} \geq 0, 1 \leq i, j \leq k$.

FKG inequalities

- The concept of FKG inequalities, which is connected with statistical mechanics and percolation theory, is related to association. It started from the works of Harris (1965), Fortuin, Kastelyn and Ginibre (1971), Holley (1974), Preston (1974), Batty (1976), Kemperman (1977) and Newman (1983). For the relationship between the two concepts, see Karlin and Rinott (1988) and Newman (1984). They observed the following - a version of the FKG inequality is equivalent to

$$\frac{\partial^2}{\partial x_i \partial x_j} \log f \geq 0 \text{ for } i \neq j \quad 1 \leq i, j \leq n$$

when $f(x_1, ..., x_n)$, the joint density of $X_1, ..., X_n$, is strictly positive on R^n.

FKG inequalities

- This is a sufficient but not a necessary condition for association of $(X_1, X_2, \ldots, X_n)$. For example, if (X_1, X_2) is a bivariate normal vector whose covariance matrix Σ is not the inverse of a matrix with non-positive off diagonal entries, then (X_1, X_2) is associated (Pitt (1982)) but the density of (X_1, X_2) does not satisfy the condition stated above.

Negative Association

- The concept of negative association as introduced by Joag-Dev and Proschan (1983) is not a dual of the theory and applications of (positive) association, but differs in several aspects.
  Definition : A set of random variables $X_1, ..., X_n$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, ..., n\}$

  $$\mathrm{Cov}(h(X_i, i \in A_1),\ g(X_j, j \in A_2)) \leq 0, \tag{6}$$

  whenever h and g are non-decreasing coordinate-wise.

Negative Association

- **Examples :** A set of independent random variables is negatively associated. Other examples of multivariate distributions that are negatively associated are components of a (a) multinomial (b) multivariate hypergeometric (c) Dirichlet and (d) Dirichlet compound multinomial distributions. However, the most interesting case is that of models of categorical data analysis where negative association (NA) and (positive) association exist side by side.

Negative Association

- Consider a model where the individuals are classified according to two characteristics. Suppose the marginal totals are fixed. Then, the marginal distributions of row (column) vectors possess NA, and the marginal distribution of a set of cell frequencies such that no pair of cells is in the same row or in the same column (for example, the diagonal cells) are (positively) associated.

Some Probabilistic Properties for Associated Sequences

- Esary, Proschan and Walkup (1967) studied the fundamental properties of association. They showed that the association of random variables is preserved under some operations, for instance,
  (i) any subset of associated random variables is associated;
  (ii) union of two independent sets of associated random variables is a set of associated random variables;
  (iii) a set consisting of a single random variable is associated;
  (iv) nondecreasing functions of associated random variables are associated; and

Some Probabilistic Properties for Associated Sequences

- (v) if $X_1^{(k)}, \ldots, X_n^{(k)}$ are associated for each k , and $\underline{X}^{(k)} = (X_1^{(k)}, \ldots, X_n^{(k)}) \to \underline{X} = (X_1, \ldots, X_n)$ in distribution, then $X_1, \ldots, X_n$ are associated.

Some Probabilistic Properties for Associated Sequences

- Esary et al (1967) have also developed a simple criterion for establishing association. Instead of checking the condition for arbitrary nondecreasing functions h and g, one can restrict to nondecreasing test functions h and g which are binary or functions h and g which are nondecreasing, bounded and continuous. In addition, they obtained bounds for the joint distribution function of associated random variables in terms of the joint distribution function of the components under independence.

Some Probabilistic Properties for Associated Sequences

- **Theorem :** If $X_1, ..., X_n$ are associated random variables, then

$$P[X_i > x_i, i = 1, ..., n] \geq \prod_{i=1}^{n} P[X_i > x_i],$$

and

$$P[X_i \leq x_i, i = 1, ..., n] \geq \prod_{i=1}^{n} P[X_i \leq x_i]. \tag{7}$$

Some Probabilistic Properties for Associated Sequences

- The concept of association is also useful in the study of approximate independence. This follows from a basic distribution function inequality due to Lebo witz (1972). Define, for A and B, subsets of $\{1, 2, ..., n\}$ and real x_j's,

$$\begin{aligned} H_{A,B} &= P[X_j > x_j; j \in A \cup B] \\ &\quad - P[X_k > x_k, k \in A] P[X_\ell > x_\ell, \ell \in B]. \end{aligned} \tag{8}$$

  Observe that the function $H(x, y)$ defined by (2) is a special case of this definition.

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Lebowitz (1972)) If $X_j, 1 \leq j \leq n$, are associated, then

$$0 \leq H_{A,B} \leq \sum_{i \in A} \sum_{j \in B} H_{\{i\},\{j\}}. \tag{9}$$

Some Probabilistic Properties for Associated Sequences

- **Proof** : Let $Z_i = I(X_i \geq x_i)$. Define
$$U(A) = \prod_{i \in A} Z_i, \text{ and } V(A) = \sum_{i \in A} Z_i.$$
Then
$$H_{A,B} = Cov(U(A), U(B)),$$
and
$$\text{Cov}(V(A), V(B)) = \sum_{i \in A} \sum_{j \in B} H_{\{i\},\{j\}}.$$
Observe that $V(A) - U(A)$ and $V(B)$ are increasing functions of Z_i, $1 \leq i \leq n$. Since Z_i's are associated, it follows that
$$\text{Cov}(V(A) - U(A), V(B)) \geq 0.$$

Some Probabilistic Properties for Associated Sequences

- Similarly, $V(B) - U(B)$ and $U(A)$ are increasing functions of $Z_i, \ 1 \leq i \leq n$ and

  $$\text{Cov}(V(B) - U(B), U(A)) \geq 0.$$

  Hence

  $$\text{Cov}(U(A), U(B)) \leq \text{Cov}(U(A), V(B)) \leq \text{Cov}(V(A), V(B)).$$

Some Probabilistic Properties for Associated Sequences

- As an immediate consequence of the Lebovitz inequality, we have the following result.

  Theorem : (Joag-Dev (1983), Newman (1984)). Suppose that $X_1, ..., X_n$ are associated. Then, $\{X_k, k \in A\}$ is independent of $\{X_j, j \in B\}$ if and only if $\text{Cov}(X_k, X_j) = 0$ for all $k \in A, j \in B$ and X_j's are jointly independent if and only if $\text{Cov}(X_k, X_j) = 0$ for all $k \neq j,\ 1 \leq k, j \leq n$. Thus, uncorrelated associated random variables are independent.

Some Probabilistic Properties for Associated Sequences

- Another fundamental inequality which is useful in proving several probabilistic results involving associated random variables is given below.

  Theorem : (Newman (1980)) Let (X, Y) be associated random variables with finite variance. Then, for any two differentiable functions h and g,

  $$|\text{Cov}(h(X), g(Y))| \leq \sup_x |h'(x)| \ \sup_y |g'(y)| \ \text{Cov}(X, Y) \tag{10}$$

  where h' and g' denote the derivatives of h and g, respectively.

  Proof is an immediate consequence of (3).

Some Probabilistic Properties for Associated Sequences

- Using the above inequality, we can prove that

$$|\mathrm{Cov}(\exp(irX), \exp(isY))| \leq |r||s|\mathrm{Cov}(X, Y) \qquad (11)$$

for $-\infty < r, s < \infty$.

This leads to the following inequality for characteristic functions of associated random sequences.

Theorem : (Newman and Wright (1981)) Suppose $X_1, ..., X_n$ are associated random variables with the joint and the marginal characteristic functions $\phi(r_1, ..., r_n)$ and $\phi_j(r_j), 1 \leq j \leq n$, respectively. Then

$$|\phi(r_1, ..., r_n) - \prod_{j=1}^{n} \phi_j(r_j)| \leq \frac{1}{2} \sum_{j \neq k} \sum |r_j|\,|r_k| \mathrm{Cov}(X_j, X_k). \quad (12)$$

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Bagai and Prakasa Rao(1991)) Suppose X and Y are associated random variables with bounded continuous densities f_X and f_Y, respectively. Then there exists an absolute constant $C > 0$ such that

$$\sup_{x,y} |P[X \leq x, Y \leq y] - P[X \leq x]P[Y \leq y]| \leq C\{\max(\sup_x f_X(x), \sup_x f_Y(x))\}^{2/3}(\mathrm{Cov}(X, Y))^{1/3}. \tag{13}$$

Some Probabilistic Properties for Associated Sequences

- The covariance structure of an associated sequence $\{X_n, n \geq 1\}$ plays a significant role in studying the probabilistic process of the associated sequence $\{X_n, n \geq 1\}$. Let

$$u(n) = \sup_{k \geq 1} \sum_{j:|j-k| \geq n} \text{Cov}(X_j, X_k), \qquad n \geq 0. \tag{14}$$

  Then, for any stationary associated sequence $\{X_j\}$, the sequence $u(n)$ is given by

$$u(n) = 2 \sum_{j=n+1}^{\infty} \text{Cov}(X_1, X_j).$$

Some Probabilistic Properties for Associated Sequences

- **Moment Bounds**
  Birkel (1988) observed that moment bounds for partial sums of associated sequences also depend on the rate of decrease of $u(n)$.

  Theorem : (Birkel (1988)) Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_j = 0,\ j \geq 1$ and suppose that

  $$\sup_{j\geq 1} E|X_j|^{r+\delta} < \infty \text{ for some } \ r > 2, \delta > 0.$$

  Assume that

  $$u(n) = O(n^{-(r-2)(r+\delta)/2\delta}).$$

Some Probabilistic Properties for Associated Sequences

- Then, there is a constant $B > 0$ not depending on n such that for all $n \geq 1$,

$$\sup_{m \geq 0} E|S_{n+m} - S_m|^r \leq Bn^{r/2} \tag{15}$$

where $S_n = \sum_{j=1}^{n} X_j$.

Some Probabilistic Properties for Associated Sequences

- If the X_j's are uniformly bounded, then the following result holds.

  Theorem : (Birkel(1988)) Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables satisfying $EX_j = 0$ and $|X_j| \leq c < \infty$ for $j \geq 1$. Assume that

  $$u(n) = O(n^{-(r-2)/2}).$$

  Then the inequality (15) holds.

Some Probabilistic Properties for Associated Sequences

- **Strong Law of Large Numbers**
  Strong law of large numbers for associated sequences have been obtained by Newman (1984) and Birkel (1989), former for the stationary case and the latter for the non-stationary case.

  Theorem : (Newman (1984)) Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. If

$$\frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}(X_1, X_j) \rightarrow 0 \text{ as } n \rightarrow \infty ,$$

  then

$$\frac{1}{n}(S_n - E(S_n)) \rightarrow 0 \text{ a.s. as } n \rightarrow \infty. \qquad (16)$$

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Birkel (1989)) Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite variance. Assume that
$$\sum_{j=1}^{\infty} \frac{1}{j^2} \text{Cov}(X_j, S_j) < \infty$$
Then the relation (16) holds.

Some Probabilistic Properties for Associated Sequences

- **Central Limit Theorem**

  Theorem : (Newman (1980, 1984)) Let $\{X_n, n \geq 1\}$ be a stationary associated sequence of random variables with

  $$E[X_1^2] < \infty \text{ and } 0 < \sigma^2 = V(X_1) + 2\sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty.$$

  Then, $n^{-1/2}(S_n - E(S_n)) \xrightarrow{\mathcal{L}} N(0, \sigma^2)$ as $n \to \infty$.

Some Probabilistic Properties for Associated Sequences

- Since, σ is not known in practice and needs to be estimated. Peligrad and Suresh (1995) obtained a consistent estimator of σ . Let $\{l_n, n \geq 1\}$ be a sequence of positive integers with $1 \leq l_n \leq n$. Set $S_j(k) = \sum_{i=j+1}^{j+k} X_i,\ B_n = \frac{1}{n-l}[\sum_{j=0}^{n-l} \frac{|S_j(l) - l_n S_n|}{\sqrt{l}}]$.

Some Probabilistic Properties for Associated Sequences

- **Theorem :** Let $\{X_n, n \geq 1\}$ be a stationary associated sequence of random variables satisfying $E(X_1) = \mu,\ E(X_1^2) < \infty$. Let $l_n = o(n)$ as $n \to \infty$. Assume that $\sum_{i=2}^{\infty} \text{Cov}(X_1, X_i) < \infty$. Then

$$B_n \to \sigma\sqrt{2/\pi} \text{ in } L_2 \text{ as } n \to \infty.$$

In addition, if we assume that $l_n = O(n/(\log n)^2)$ as $n \to \infty$, the convergence above is almost sure.

Some Probabilistic Properties for Associated Sequences

- Cox and Grimmett (1984) proved a Central limit theorem for double sequences and used it in percolation theory and voter model.

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Cox and Grimmett (1984)) Let $\{X_{nj}, 1 \leq j \leq n, n \geq 1\}$ be a triangular array of associated random variables satisfying:
  (i) there are strictly positive, finite constants c_1, c_2 such that

  $$\text{Var}(X_{nj}) \geq c_1, E[|X_{nj}|^3] \leq c_2 \quad \forall \quad j \text{ and } n.$$

  (ii) there is a function $u : \{0, 1, 2, ...,\} \to \mathcal{R}$ such that $u(r) \to 0$ as $r \to \infty$ and

  $$\sum_{j:|k-j|\geq r} \text{Cov}(X_{nj}, X_{nk}) \leq u(r) \text{ for all } k, n \text{ and } r \geq 0.$$

  Let $S_{n,n} = \sum_{j=1}^{n} X_{nj}$. Then the sequence $\{S_{n,n}, n \geq 1\}$ satisfies the central limit theorem.

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Birkel (1988)) Let $\{X_n, n \geq 1\}$ be an associated sequence with $E[X_n] = 0$, satisfying
  (i) $u(n) = O(e^{-\lambda n}), \quad \lambda \geq 0,$
  (ii) $\inf\limits_{n \geq 1} \dfrac{\sigma_n^2}{n} > 0, \quad \sigma_n^2 = E[S_n^2],$ and
  (iii) $\sup\limits_{n \geq 1} E[|X_n|^3] < \infty$. where $u(n)$ is as defined earlier.

Some Probabilistic Properties for Associated Sequences

- Then there exists a constant B not depending on n such that for all $n \geq 1$,

$$\Delta_n \equiv \sup_{x \in R} |P\{\sigma_n^{-1} S_n \leq x\} - \Phi(x)| \leq Bn^{-1/2} \log^2 n.$$

If, instead of (iii), we assume that

$$\sup_{j \geq 1} E|X_j|^{3+\delta} < \infty \quad \text{for some} \quad \delta > 0,$$

then there exists a constant B not depending on n such that for all $n \geq 1$,

$$\Delta_n \leq Bn^{-1/2} \log n.$$

Some Probabilistic Properties for Associated Sequences

- **Theorem :** (Dewan and Prakasa Rao (1997)) Let $\{X_i, 1 \leq i \leq n\}$ be a set of stationary associated random variables with $E[X_1] = 0$, $\text{Var}[X_1] = \sigma_0^2 > 0$ and $E[|X_1|^3] < \infty$. Suppose the distribution of X_1 is absolutely continuous. Let $S_n = \sum_{i=1}^{n} X_i$ and $\sigma_n^2 = \text{Var}(S_n)$. Suppose that $\frac{\sigma_n^2}{n} \to \sigma_0^2$ as $n \to \infty$. Let $F_n(x)$ be the distribution function of $\frac{S_n}{\sigma_n}$ and $F_n^*(.)$ be the distribution function of $\frac{\sum_{i=1}^{n} Z_i}{\sigma_n}$ where $Z_i, 1 \leq i \leq n$ are i.i.d. with distribution function same as that of X_1. Let m_n be a bound on the derivative of F_n^* .

Some Probabilistic Properties for Associated Sequences

- Then there exist absolute constants $B_i > 0$, $1 \leq i \leq 3$, such that

$$\sup_x |F_n(x) - \Phi(x)| \leq B_1 \frac{{d_n}^{1/3} {m_n}^{2/3}}{{\sigma_n}^{2/3}} + B_2 \frac{E|X_1|^3}{\sqrt{n}\, \sigma_0^3} + B_3 \left(\frac{\sigma_n}{\sigma_0 \sqrt{n}} - 1\right) \quad (17)$$

where

$$d_n = \sum_{j=2}^{n} (n - j + 1) \text{Cov}\,(X_1, X_j).$$

Demimartingales

- Newman and Wright (1982) introduced the notion of demimartingales and proved some submartingale-type inequalities for associated random variables.

Definition : A sequence of random variables $S_1, S_2, \ldots$ in L^1 is called a demimartingale if for $j = 1, 2, \ldots,$ and all coordinate-wise nondecreasing functions g

$$E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \geq 0, \quad (18)$$

provided the expectation is defined. If the inequality (18) is modified so that g is required to be non-negative (resp., non-positive) and non-decreasing, then the sequence will be called a demisubmartingale (resp., demisupermartingale).

Demimartingales

- **Remark :** If $X_1, X_2, \ldots,$ are L^1, mean zero associated sequence of random variables and $S_j = X_1 + \ldots + X_j$ with $S_0 = 0$, then the sequence $\{S_n, n \geq 1\}$ is a demimartingale.

Demimartingales

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose that a set of random variables $X_1, \ldots, X_n$ defined on the probability space $(\Omega, \mathcal{F}, P)$ have mean zero and are associated. Let $S_0 = 0$ and $S_j = X_1 + \cdots + X_j, j = 1, \ldots, n$. Then it follows that, for any componentwise nondecreasing function g,

  $$E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \geq 0, j = 1, \ldots, n \tag{19}$$

  provided the expectation exists.

Demimartingales

- Recall that a sequence of random variables $\{S_n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a martingale with respect to the natural sequence $\mathcal{F}_n = \sigma\{S_1, \ldots, S_n\}$ of σ-algebras if $E(S_{n+1}|S_1, \ldots, S_n) = S_n$ a.s. for $n \geq 1$. Here $\sigma\{S_1, \ldots, S_n\}$ denotes the σ-algebra generated by the random sequence $S_1, \ldots, S_n$. An alternate way of defining the martingale property of the sequence $\{S_n, n \geq 1\}$ is that

$$E((S_{n+1} - S_n)g(S_1, \ldots, S_n)) = 0, n \geq 1$$

for all measurable functions $g(x_1, \ldots, x_n)$ assuming that the expectations exist.

Demimartingales

- Newman and Wright (1982) introduced the notion of demimartingales.

  Definition : A sequence of random variables $\{S_n, n \geq 1\}$ in $L^1(\Omega, \mathcal{F}, P)$ is called a *demimartingale* if, for every componentwise nondecreasing functions g,

$$E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \geq 0, j \geq 1 \tag{20}$$

Demimartingales

- **Remarks :** If $X_1, X_2, \ldots,$ are L^1, mean zero associated random variables and $S_j = X_1 + \ldots + X_j$ with $S_0 = 0$, then $S_1, S_2, \ldots$ is a demimartingale.

  If g is required to be nonnegative (resp., non-positive) and nondecreasing in (20), then the sequence will be called a *demisubmartingale* (resp., *demisupermartingale*).

Demimartingales

- A square integrable martingale $\{S_n, \mathcal{F}_n, n \geq 1\}$ with the natural choice of σ-algebras $\{\mathcal{F}_n, n \geq 1\}, \mathcal{F}_n = \sigma\{S_1, \ldots, S_n\}$ is a demimartingale. This can be seen by noting that

$$\begin{aligned} & E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \\ &= E[E((S_{j+1} - S_j)g(S_1, \ldots, S_j)|\mathcal{F}_j)] \\ &= E[g(S_1, \ldots, S_j)E((S_{j+1} - S_j)|\mathcal{F}_j)] = 0 \end{aligned} \tag{21}$$

by the martingale property of the process $\{S_n, \mathcal{F}_n, n \geq 1\}$.

Demimartingales

- Similarly it can be seen that every submartingale $\{S_n, \mathcal{F}_n, n \geq 1\}$ with the natural choice of σ-algebras $\{\mathcal{F}_n, n \geq 1\}, \mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ is a demisubmartingale.

Demimartingales

- However a demisubmartingale need not be a submartingale. This can be seen by the following example (cf. Hadjikyriakou (2010)).

  Example : Let the random variables $\{X_1, X_2\}$ be such that

  $P(X_1 = -1, X_2 = -2) = p$ and $P(X_1 = 1, X_2 = 2) = 1 - p$

  where $0 \leq p \leq \frac{1}{2}$. Then the finite sequence $\{X_1, X_2\}$ is a demisubmartingale since for every nonnegative nondecreasing function $g(.)$,

$$
\begin{aligned}
E[(X_2 - X_1)g(X_1)] &= -pg(-1) + (1-p)g(1) \qquad (22)\\
&\geq -pg(-1) + pg(1) \ \ (\text{since } \ p \leq \frac{1}{2})\\
&= p(g(1) - g(-1)) \geq 0.
\end{aligned}
$$

Demimartingales

- However the sequence $\{X_1, X_2\}$ is not a submartingale since

$$E(X_2|X_1 = -1) = \sum_{x_2=-2,2} x_2 P(X_2 = x_2|X_1 = -1) = -2 < -1.$$

Demimartingales

- As remarked earlier, the sequence of partial sums of mean zero associated random variables is a demimartingale. However the converse need not hold. In other words, there exist demimartingales such that the demimartingale differences are not associated.

Demimartingales

- This can be seen again by the following example (cf. Hadjikyriakou (2010)).

Example : Let X_1 and X_2 be random variables such that

$$P(X_1 = 5, X_2 = 7) = \frac{3}{8}, P(X_1 = -3, X_2 = 7) = \frac{1}{8}$$

and

$$P(X_1 = -3, X_2 = 7) = \frac{4}{8}.$$

Let g be a nondecreasing function. Then the finite sequence $\{X_1, X_2\}$ is a demimartingale since

$$E[(X_2 - X_1)g(X_1)] = \frac{6}{8}[g(5) - g(-3)] \geq 0.$$

Let f be a nondecreasing function such that

$$f(x) = 0 \text{ for } x < 0, f(2) = 2, f(5) = 5 \text{ and } f(10) = 20.$$

Demimartingales

- Christofides (2004) gave another method of constructing a demimartingale through U-Statistics.

Demimartingales

- **Example :** Let $X_1, \ldots, X_n$ be associated random variables and let $h(x_1, \ldots, x_m)$ be a "kernel" mapping R^m to R where $1 \leq m \leq n$. Without loss of generality, we assume that h is a symmetric function. Define the U-statistic

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m}) \tag{23}$$

where $\sum_{1 \leq i_1 < \cdots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m})$ denotes the summation over the $\binom{n}{m}$ combinations of m distinct elements $\{i_1, \ldots, i_m\}$ from $\{1, \ldots, n\}$. Suppose that the function h is componentwise nondecreasing and $E(h) = 0$. Then the sequence $\{S_n = \binom{n}{m} U_n, n \geq m\}$ is a demimartingale.

Demimartingales

- Another method of construction of demimartingales is given by the following result.
  Theorem : (Christofides (2000)) Let the random sequence $\{S_n, n \geq 1\}$ be a demisubmartingale (or a demimartingale) and g be a nondecreasing convex function. Then the random sequence $\{g(S_n), n \geq 1\}$ is a demisubmartingale.

Demimartingales

- As an application of this theorem, we have the following result.
  Theorem : If $\{S_n, n \geq 1\}$ is a demimartingale, then $\{S_n^+, n \geq 1\}$ is a demisubmartingale and $\{S_n^-, n \geq 1\}$ is also a demisubmartingale.

  Proof: Since the function $g(x) \equiv x^+ = \max(0, x)$ is nondecreasing and convex, it follows that $\{S_n^+, n \geq 1\}$ is a demisubmartingale from the previous theorem. Let $Y_n = -S_n, n \geq 1\}$. It is easy to see that $\{Y_n, n \geq 1\}$ is also a demimartingale and $Y_n^+ = S_n^-$ where $x^- = \max(0, -x)$. Hence, as an application of the first part of the theorem, it follows that the sequence $\{S_n^-, n \geq 1\}$ is a demisubmartingale.

Maximal inequality for Demimartingales

- The next result is a Doob type maximal inequality for demisubmartingales due to Newman and Wright (1982). Let

$$S_n^* = \max(S_1, \ldots, S_n). \tag{24}$$

Define the rank orders $R_{n,j}$ by

$$R_{n,j} = \begin{cases} j-th \text{ largest of } (S_1, \ldots, S_n), & \text{if } j \leq n, \\ \min(S_1, \ldots, S_n) = R_{n,n}, & \text{if } j > n, \end{cases}$$

Demimartingales

- **Theorem :** Suppose $S_1, S_2, \ldots$ is a demimartingale (resp., demisubmartingale) and m is a nondecreasing (resp., nonnegative and nondecreasing) function on $(-\infty, \infty)$ with $m(0) = 0$; then, for any n and j,

$$E(\int_0^{R_{n,j}} u\, dm(u)) \leq E(S_n m(R_{n,j})); \tag{25}$$

and, for any $\lambda > 0$,

$$\lambda\, P(R_{n,j} \geq \lambda) \leq \int_{[R_{n,j} \geq \lambda]} S_n dP. \tag{26}$$

Maximal Inequality for Demimartingales

- **Proof :** For fixed n and j, let $Y_k = R_{k,j}$ and $Y_0 = 0$. Then

$$S_n m(Y_n) = \sum_{k=0}^{n-1} S_{k+1}(m(Y_{k+1}) - m(Y_k)) + \sum_{k=1}^{n-1}(S_{k+1} - S_k)m(Y_k). \tag{27}$$

From the definition of $R_{n,j}$, it follows that
for $k < j$, either $Y_k = Y_{k+1}$ or $S_{k+1} = Y_{k+1}$;
and
for $k \geq j$, either $Y_k = Y_{k+1}$ or $S_{k+1} \geq Y_{k+1}$.
Hence, for any k,

$$\begin{aligned} S_{k+1}(m(Y_{k+1} - m(Y_k)) &\geq Y_{k+1}(m(Y_{k+1} - m(Y_k)) \quad (28) \\ &\geq \int_{Y_k}^{Y_{k+1}} u \;\; dm(u). \end{aligned}$$

Demimartingales

- Combining the above inequalities , it follows that

$$S_n m(Y_n) \geq \int_0^{Y_n} u \ \ dm(u) + \sum_{k=1}^{n-1} (S_{k+1} - S_k) m(Y_k). \quad (29)$$

Note that

$$E[(S_{k+1} - S_k) m(Y_k)] \geq 0, 1 \leq k \leq n-1 \quad (30)$$

by the definition of demimartingale (resp., demisubmartingale) since the function $m(Y_k)$ is a nondecreasing (resp., nonnegative and nondecreasing) function of $S_1, \ldots, S_k$. Taking the expectations on both sides of the inequality (29) and applying the inequality (30), we obtain the inequality stated in the theorem by observing that $Y_n = R_{n,j}.$. The inequality (26) is an easy consequence by choosing the function $m(u)$ to be the indicator function $I_{[u \geq \lambda]}$.

An Upcrossing Inequality for Demisubmartingales

- The following theorem extends Doob's upcrossing inequality for submartingales to demisubmartingales. Given the sequence $S_1, S_2, \ldots, S_n$ and $a < b$, we define a sequence of stopping times $J_0 = 0, J_1, J_2, \ldots$ as follows (for $k = 1, 2, \ldots$):

$$J_{2k-1} = \begin{cases} n+1 & \text{if } \{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\} \text{ is empty} \\ \min\{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\}, & \text{otherwise,} \end{cases}$$

and

$$J_{2k-1} = \begin{cases} n+1 & \text{if } \{j : J_{2k-2} < j \leq n \text{ and } S_j \geq b\} \text{ is empty} \\ \min\{j : J_{2k-2} < j \leq n \text{ and } S_j \geq b\}, & \text{otherwise.} \end{cases}$$

The number of complete upcrossings of the interval $[a, b]$ by $S_1, \ldots, S_n$ is denoted by $U_{a,b}$ where

$$U_{a,b} = \max\{k : J_{2k} < n+1\}. \quad (31)$$

Demimartingales

- **Theorem:** If the finite sequence $\{S_1, \ldots, S_n\}$ is a demisubmartingale, then, for $a < b$,

$$E(U_{a,b}) \leq \frac{E((S_n - a)^+) - E((S_1 - a)^+)}{b - a}.$$

Demimartingales

- The next theorem gives sufficient conditions for the almost sure convergence of a demimartingale. It is a consequence of the upcrossing inequality as in the case of martingales.

  Theorem : If $\{S_n\}$ is a demimartingale and $\limsup_{n\to\infty} E|S_n| < \infty$, then S_n converges a.s. to a random variable X such that $E|X| < \infty$.

Whittle Type Inequality for Demimartingales

- We now discuss a Whittle type inequality for demisubmartingales due to Prakasa Rao (2002). This result generalizes the Kolmogorov inequality and the Hajek-Renyi inequality for independent random variables (Whittle (1969)) and is an extension of the results in Christofides (2000) for demisubmartingales.

  Let $\{S_n, n \geq 1\}$ be a demisubmartingale. Suppose $\phi(.)$ is a nondecreasing convex function. Then the sequence $\{\phi(S_n), n \geq 1\}$ is a demisubmartingale (cf. Christofides (2000)).

Whittle Type Inequality for Demimartingales

- **Theorem :** Let $S_0 = 0$ and suppose the sequence of random variables $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $\phi(.)$ be nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Let A_n be the event that $\phi(S_k) \leq \psi(u_k), 1 \leq k \leq n$, where $0 = u_0 < u_1 \leq \cdots \leq u_n$.. Then

$$P(A_n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}. \tag{32}$$

Random Generation of Associated Sequences

- Matula (1996) discusses two methods which can be used for generating associated sequences.

  (i) Let $\{Y_n, n \geq 1\}$ be a sequence of independent and identically distributed standard normal random variables . Let

$$X_n = I_{(-\infty,u)}\left(\frac{Y_1 + Y_2 + \ldots + Y_n}{\sqrt{n}}\right) \text{ for arbitrary } u \in R.$$

  Then, $\{X_n, n \geq 1\}$ is a sequence of associated random variables with

$$\text{Cov}(X_j, X_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp[-\frac{x^2}{2}](\Phi(\frac{\sqrt{n}u - \sqrt{j}x}{\sqrt{n-j}}) - \Phi(u))dx,$$

  for $j < n$, where Φ denotes the standard normal distribution function.

Random Generation of Associated Sequences

- (ii) Let $\{Y_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $E(Y_1) = 1$ and $E(Y_1^2) = 2$. For $n \geq 1$, let

$$X_n = \frac{1}{2^{n-1}} Y_1 + \ldots + \frac{1}{2^{n-1}} Y_{n-1} + nY_n.$$

Then, $\{X_n, n \geq 1\}$ is a sequence of associated random variables with

$$\mathrm{Cov}(X_j, X_n) = \frac{1}{2^{n-1}}(j + \frac{j-1}{2^{j-1}})$$

for $j < n$.

Random Generation of Associated Sequences

- Reference: "Associated Sequences, Demimartingales and Nonparametric Inference", B.L.S. Prakasa Rao, Birkhauser, Springer, Basel (2012).