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MAXIMAL INEQUALITIES FOR FRACTIONAL BROWNIAN MOTION: AN OVERVIEW

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Abstract: We give an overview of some maximal inequalities and limit theorems for the tail probabilities for the supremum of a fractional Brownian motion.

1 Introduction

Fractional Brownian motion plays an important role in modelling long range dependence in time series analysis. Developing methods of statistical inference for fractional diffusion processes, that is, for processes driven by a fractional Brownian motion (fBm) is of importance and interest. Even though there is a vast amount of literature dealing with Gaussian processes, including an excellent survey in Li and Shao (2001), we have decided to prepare an overview of the results, especially for the case of fractional Brownian motion, for the use of developers of statistical methods in their study of long range dependence. Maximal inequalities and limit theorems for fBm plays an important role in such developments. For a recent survey on mthods of statistical inference for fractional diffusion processes, see Prakasa Rao (2010).

Let $B^H \equiv \{B_t^H, -\infty < t < \infty\}$ be the standard fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, that is, a centered Gaussian process with $B_0^H = 0$ and

$$Cov(B_s^H, B_t^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), -\infty < s, t < \infty.$$

It is known that the fBm B^H is self-similar, that is, for any $c > 0$,

$$\{B_{ct}^H, -\infty < t < \infty\} \stackrel{\Delta}{=} \{c^H B_t^H, -\infty < t < \infty\}$$

in the sense that the processes specified on both sides have the same finite dimensional distributions and it has stationary Gaussian increments. For a short survey of properties of the fBm, see Prakasa Rao (2010).

The following result is due to Ruzmaikina (2000). Let $H \in (\frac{1}{2}, 1)$ and $0 < \beta < 2H - 1$.

Let a function $f(.) \geq 0$ and suppose that $f() \in L^{\frac{2}{1+\beta}}([0,1])$. Define

$$q_f(s,t) = H(2H-1)\int_s^t \int_s^t f(u)f(v)|u-v|^{2H-2}\, dudv. \tag{1. 1}$$

Theorem 1.1: (Ruzmaikina (2000)) Let $H \in (\frac{1}{2}, 1)$ and $0 < \beta < 2H-1$. Let f be a function such that $f(.) \geq 0$ and $f(.) \in L^{\frac{2}{1+\beta}}([0,1])$. Then there exists a Gaussian Markov process $\{Y(t), t \geq 0\}$ with independent increments and continuous paths such that $E[Y(t)] = 0, t \geq 0$ and

$$E[Y(s)Y(t)] = q_f(0,s) = H(2H-1)\int_0^s \int_0^s f(u)f(v)|u-v|^{2H-2}\, dudv, 0 \leq s \leq t.$$

Proof: Note that the function $q_f(0,s)$ is non-decreasing in s since the function f is non-negative. Furthermore it is nonnegative definite. This can be seen by the following arguments. Let $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n$ and $c_1, \ldots, c_n \in R$. Then

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n c_i c_j q_f(0,s_i) &= \sum_{i=1}^n c_i^2 q_f(0,s_i) + \sum_{i=1}^n (2\sum_{i<j} c_i c_j) q_f(0,s_i) \\
&= \sum_{i=1}^n [c_i^2 + 2\sum_{i<j} c_i c_j] q_f(0,s_i) \\
&= \sum_{i=1}^n [(c_i + \ldots + c_n)^2 - (c_{i+1} + \ldots + c_n)^2] q_f(0,s_i) \\
&= (c_1 + \ldots + c_n)^2 q_f(0,s_1) + \sum_{i=2}^n (c_i + \ldots + c_n)^2 [q_f(0,s_{i+1}) - q_f(0,s_i)) \\
&\geq 0.
\end{aligned}$$

Hence one can construct a mean zero Gaussian process with covariance function $E[Y(s)Y(t)] = q_f(0,s), 0 \leq s \leq t \leq 1$. Note that, for $0 \leq s_1 < t_1 \leq s_2 < t_2$,

$$\begin{aligned}
&E[(Y(t_1) - Y(s_1))(Y(t_2) - Y(s_2))] \\
&= E[Y(t_1)Y(t_2)] - E[Y(t_1)Y(s_2)] \\
&\quad -E[Y(s_1)Y(t_2)] + E[Y(s_1)Y(s_2)] \\
&= q_f(0,t_1) - q_f(0,t_1) - q_f(0,s_1) + q_f(0,s_1) \\
&= 0.
\end{aligned}$$

Hence the Gaussian process $\{Y(t), t \geq 0\}$ has uncorrelated increments. Since the increments of the process $\{Y(t), t \geq 0\}$ are Gaussian and they are uncorrelated, they have to be independent. This shows that the process $\{Y(t), t \geq 0\}$ is a mean zero Gaussian Markov process with independent increments.$\diamondsuit$.

The proof given above is due to Ruzmaikina (2000). We included it here for completeness.

The process $\{Y(t), t \geq 0\}$ described above can be represented as the deterministic time change of the standard Brownian motion $\{W(t), t \geq 0\}$; in fact the process $\{Y(t), t \geq 0\}$ and the process $\{W(q_f(0,t)), t \geq 0\}$ have the same probability structure. In particular, the process $\{Y(t), t \geq 0\}$ has continuous sample paths with probability one and it obeys the reflection principle in the sense that

$$P(\sup_{0\leq t\leq T} Y(t) \geq \lambda) = 2\, P(Y(T) \geq \lambda) \tag{1. 2}$$

for every $\lambda > 0$ and $T > 0$. Hence

$$P(\sup_{0\leq t\leq T} Y(t) \geq \lambda) = \int_{\lambda/\sqrt{q_f(0,T)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx. \tag{1. 3}$$

The following result is due to Slepian (1962) and is known as the Slepian's lemma (cf. Kahane (1986) ; Leadbetter et al. (1983); Adler (1990)).

Theorem 1.2: (Slepian's Lemma) Let the processes $X_1 = \{X_1(t), t \geq 0\}$ and $X_2 = \{X_2(t), t \geq 0\}$ be centered Gaussian processes with $E[X_1^2(t)] = E[X_2^2(t)] = 1$. Let $\rho_1(t,s)$ and $\rho_2(t,s)$ be the covariance functions of the processes X_1 and X_2 respectively. Suppose that, for some $\delta > 0$,

$$\rho_1(t,s) \geq \rho_2(t,s), 0 \leq t, s \leq \delta.$$

Then

$$P(\sup_{0\leq t\leq T} X_1(t) \leq u) \geq P(\sup_{0\leq t\leq T} X_2(t) \leq u), u \in R$$

for any $0 \leq T \leq \delta$.

Note that the function $q_f(0,t) = t^{2H}$ for the function $f \equiv 1$. Theorem 1.1 implies that there exists a centered Gaussian Markov process $\hat{B}^H = \{\hat{B}_t^H, 0 \leq t \leq T\}$ with independent increments such that

$$Cov(\hat{B}_s^H, \hat{B}_t^H) = s^{2H}, 0 \leq s \leq t \leq T.$$

Furthermore the "reflection principle" holds for the process $\hat{B}^H$ in the sense that, for all $u > 0$, and $T > 0$,

$$P(\sup_{0\leq t\leq T} \hat{B}_t^H \geq u) = 2\, P(\hat{B}_T^H \geq u).$$

Note that, for any $u > 0$, by the Slepian's lemma,

$$P(\sup_{0\leq t\leq T} B_t^H \geq u) \leq P(\sup_{0\leq t\leq T} \hat{B}_t^H \geq u)$$

and hence, for any $u > 0$,

$$P(\sup_{0\leq t\leq T} B_t^H \geq u) \leq 2\, P(\hat{B}_T^H \geq u). \tag{1. 4}$$

Observe that the random variable $\hat{B}_T^H$ has the Gaussian distribution with mean zero and variance T^{2H}. Therefore

$$P(\hat{B}_T^H \geq u) = P(Z \geq uT^{-H})$$

where Z is a standard Gaussian random variable. It is known that

$$P(Z \geq u) \leq \frac{1}{2} e^{-u^2/2},$$

and

$$P(Z \geq u) \leq \frac{1}{\sqrt{2\pi}u} e^{-u^2/2}$$

for any $u > 0$ (cf. Ito and Mckean (1965), p.17; Kutoyants (1994), p.27). Applying these inequalities, we can get the following maximal inequality for the fBm B^H with Hurst index $H \in (\frac{1}{2}, 1)$.

Theorem 1.3: Suppose the process B^H is a centered fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then, for any $u > 0$,

$$P(\sup_{0\leq t\leq T} B_t^H \geq u) \leq 2 \min\{\frac{1}{2} e^{-T^{-2H}u^2/2}, \frac{1}{\sqrt{2\pi}uT^{-H}} e^{-u^2T^{-2H}/2}\}.$$

From the symmetry property of the fBm, it follows that

$$\begin{aligned} P(\sup_{0\leq t\leq T} B_t^H \geq u) &= P(-\sup_{0\leq t\leq T} B_t^H \leq -u) \\ &= P(\inf_{0\leq t\leq T} (-B_t^H) \leq -u) \\ &= P(\inf_{0\leq t\leq T} B_t^H \leq -u). \end{aligned}$$

Hence, for any $u > 0$,

$$P(\sup_{0\leq t\leq T} |B_t^H| \geq u) \leq P(\sup_{0\leq t\leq T} B_t^H \geq u)$$

$$
\begin{aligned}
& +P(\inf_{0\le t\le T} B_t^H \le -u) \\
= \;& 2\,P(\sup_{0\le t\le T} B_t^H \ge u) \\
\le \;& 2\,P(\sup_{0\le t\le T} \hat{B}_t^H \ge u) \\
= \;& 4P(\hat{B}_T^H \ge u),
\end{aligned}
$$

and we have the following result.

Theorem 1.4: Suppose the process B^H is the centered fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then, for any $u > 0$,

$$
P(\sup_{0\le t\le T} |B_t^H| \ge u) \le 4\min\{\frac{1}{2}e^{-T^{-2H}u^2/2}, \frac{1}{\sqrt{2\pi}uT^{-H}}e^{-u^2T^{-2H}/2}\}.
$$

Note that $B_0^H = 0$ a.s. and the fBm B^H is self-similar with stationary increments. Hence

$$
\begin{aligned}
\sup_{s\le t\le s+r} |B_t^H - B_s^H| \;&\triangleq\; \sup_{0\le t\le r} |B_t^H - B_0^H| \\
&\overset{a.s.}{=}\; \sup_{0\le t\le r} |B_t^H| \\
&\triangleq\; r^H \sup_{0\le t\le 1} |B_t^H|
\end{aligned}
$$

Hence, for any $u > 0$,

$$
\begin{aligned}
P(\sup_{s\le t\le s+r} |B_t^H - B_s^H| \ge u) \;&=\; P(\sup_{0\le t\le 1} |B_t^H| \ge ur^{-H}) \\
&\le\; 2\,P(\sup_{0\le t\le 1} \hat{B}_t^H \ge ur^{-H}) \le 4\,P(\hat{B}_1^H \ge ur^{-H}).
\end{aligned}
$$

Following the arguments used above, Muneya and Shieh (2009) obtained the following maximal inequalities for the fBm $\{B_t^H, t \ge 0\}$ with $B_0^H = 0$.

Theorem 1.5: Let $H \in [\frac{1}{2}, 1)$. Then, for any $u > 0$,

$$
P(\sup_{t\le s\le t+1} B_s^H \ge u) \le \sqrt{\frac{2}{\pi}} \int_u^\infty e^{-x^2/2}dx \tag{1. 5}
$$

and

$$(1.\ 6) \qquad P(\sup_{t\le s\le t+1}|B_s^H|\ge u)\le 2\sqrt{\frac{2}{\pi}}\int_u^\infty e^{-x^2/2}dx.$$

Remarks : Let the process $\{W(t), t\ge 0\}$ be the standard Brownian motion. Without taking recourse to Theorem 1.1, one can still obtain the inequalities, stated in Theorems 1.2 to Theorem 1.5, by observing that the scaled Brownian motion $\{Z(t)=W(t^{2H}), t\ge 0\}$ satisfies the conditions in the Slepian's lemma for $H\in(\frac{1}{2},1)$, and the process Z obeys the reflection principle.

Ruzmaikina (2000) obtained the following inequality for a stochastic integral with respect to a fBm.

Theorem 1.6: (Ruzmaikina (2000)) Let $H\in(\frac{1}{2},1)$ and $0<\beta<2H-1$. Suppose the function $f(.)\in L^{2/(1+\beta)}([0,1])$. Define

$$q_f(s,t)=H(2H-1)\int_s^t\int_s^t f(u)f(v)|u-v|^{2H-2}dudv.$$

Then, for any $\lambda>0$,

$$(1.\ 7) \qquad P(\sup_{0\le t\le 1}\int_0^t f(s)dB_s^H\ge\lambda) \le \int_{\lambda r/\sqrt{q_{f_+}(0,1)}}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx + \int_{\lambda(1-r)/\sqrt{q_{f_-}(0,1)}}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx$$

where $f_+=\frac{|f|+f}{2}$ and $f_-=\frac{|f|-f}{2}$.

Proof : This result is again a consequence of the Slepian's lemma and the Theorem 1.1. The following proof is due to Ruzmaikina (2000). We give the proof here for completeness. Let

$$X(t)=\int_0^t f(s)dB_s^H.$$

Then the process $\{X(t), 0\le t\le 1\}$ is a centered Gaussian process with

$$E[X(t)X(s)]=q_f(s,t), 0\le t,s\le 1.$$

Define the process $\{Y(t), 0 \leq t \leq 1\}$ to be a centered Gaussian process with

$$E[Y(t)Y(s)] = q_f(0,s), 0 \leq s \leq t \leq 1.$$

Existence of such a process was proved in the Theorem 1.1. It is clear that $E([X(t)]^2) = E([Y(t)]^2)$. Let us first assume that the function $f \geq 0$. Then $E[X(s)X(t)] \geq E[Y(s)Y(t)]$ for $0 \leq s, t \leq 1$. Hence the processes $\{X(t), 0 \leq t \leq 1\}$ and $\{Y(t), 0 \leq t \leq 1\}$ satisfy the conditions in the Slepian's lemma. Furthermore the process $\{Y(t), 0 \leq t \leq 1\}$ is a Gaussian process with independent increments and obeys the reflection principle. Hence, for any $\lambda > 0$,

$$\begin{aligned} P(\sup_{0\leq t\leq 1} X(t) \geq \lambda) &\leq P(\sup_{0\leq t\leq 1} Y(t) \geq \lambda) \\ &= 2\,P(Y(1) \geq \lambda) \\ &= \int_{\lambda r/\sqrt{q_f(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx. \end{aligned}$$

Therefore, for the function $f(.) \geq 0$,

$$(1.\ 8) \qquad P(\sup_{0\leq t\leq 1} X(t) \geq \lambda) \leq \int_{\lambda r/\sqrt{q_f(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx.$$

Suppose $f(.) \in L^{2/(1+\beta)}([0,1])$. Let

$$X_+(t) = \int_0^t f_+(s) dB_s^H, 0 \leq t \leq 1$$

and

$$X_-(t) = \int_0^t f_-(s) dB_s^H, 0 \leq t \leq 1.$$

Define the processes $\{Y_+(t), 0 \leq t \leq 1\}$ and $\{Y_-(t), 0 \leq t \leq 1\}$ by replacing the function f by f_+ and f_- respectively in the process $\{Y(t), 0 \leq t \leq 1\}$ defined earlier. Since the Slepian's lemma applies to the processes $\{X_-(t), 0 \leq t \leq 1\}$ and the process $\{Y_-(t), 0 \leq t \leq 1\}$, we get that

$$(1.\ 9) \qquad P(\sup_{0\leq t\leq 1} X_+(t) \geq \lambda) \leq \int_{\lambda r/\sqrt{q_{f_+}(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx.$$

and

$$(1.\ 10) \qquad P(\sup_{0\leq t\leq 1} X_-(t) \geq \lambda) \leq \int_{\lambda r/\sqrt{q_{f_-}(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx.$$

Therefore, for any $0 \leq r \leq 1$,

$$(1.\ 11) \quad P(\sup_{0\leq t\leq 1} X(t) \geq \lambda) = P(\sup_{0\leq t\leq 1} (X_+(t)) + \sup_{0\leq t\leq 1} (-X_-(t)) \geq \lambda)$$

$$
\begin{aligned}
&\leq P(\sup_{0\leq t\leq 1} X_+(t) \geq \lambda r) + P(\sup_{0\leq t\leq 1} (-X_-(t) \geq \lambda(1-r)) \\
&\leq \int_{\lambda r/\sqrt{q_{f_+}(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx \\
&\quad + \int_{\lambda(1-r)/\sqrt{q_{f_-}(0,1)}} \sqrt{\frac{2}{\pi}} e^{-x^2/2} dx.
\end{aligned}
$$

2 Maximal Inequalities Leading to Upper Bounds

Novikov and Valkeila (1999) obtained some maximal inequalities for fractional Brownian motion. They extend the Burkholder-Davis-Gundy inequalities for the fractional Brownian motion. Let $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by the fractional Brownian motion B^H. For any process X, let X^* be the supremum process defined by

$$X_t^* = \sup_{0\leq s\leq t} |X_s|.$$

Since the fractional Brownian motion is a self-similar process, it follows that

$$\{B_{ct}^H, 0 \leq t \leq T\} \stackrel{\Delta}{=} \{c^H B_t^H, 0 \leq t \leq T\}$$

and hence

$$B_{ct}^{*H} \stackrel{\Delta}{=} c^H B_t^{*H}$$

for any $c > 0$. The following result is an easy consequence of the self-similarity of the fractional Brownian motion B^H.

Theorem 2.1: (Novikov and Valkeila (1999)) For any $T > 0$ and $p > 0$,

$$E[(B_T^{*H})^p] = K(H,p) T^{pH} \tag{2. 1}$$

where $K(H,p) = E[(B_1^{*H})^p]$.

Novikov and Valkeila (1999) extended the above result for a fractional Brownian motion indexed by a stopping time.

Theorem 2.2: Let τ be a stopping time with respect to the natural filtration $\mathcal{F}$ generated by the fBm B^H. Then, for any $p > 0$, and $H \in (\frac{1}{2}, 1)$, there exist positive constants $c(p,H)$ and $C(p,H)$ depending only on p and H such that

$$c(p,H) E(\tau^{pH}) \leq E[(B_\tau^{*H})^p] \leq C(p,H) E(\tau^{pH}) \tag{2. 2}$$

and, for any $p > 0$, and $H \in (0, \frac{1}{2})$,

$$E[(B_\tau^{*H})^p] \geq c(p, H) E(\tau^{pH}). \tag{2. 3}$$

Muneya and Shieh (2009) obtained the following result on the moments of supremum of the fBm B^H over a bounded interval.

Theorem 2.3: (Muneya and Shieh (2009)) Let $H \in [\frac{1}{2}, 1)$. Then, for any integer $m \geq 1$,

$$E[(\sup_{t \leq s \leq t+r} |B_t^H|)^m] \leq r^{Hm} \frac{2\sqrt{2}}{\sqrt{\pi}} (m-1)!! \text{ if } m \text{ is odd} \tag{2. 4}$$

and

$$E[(\sup_{t \leq s \leq t+r} |B_t^H|)^m] \leq r^{Hm}\ 2(m-1)!! \text{ if } m \text{ is even.} \tag{2. 5}$$

This result follows from the following arguments. Note that, for any non-negative random variable X,

$$E(X^m) = \int_0^\infty m y^{m-1} P(X > y)\, dy.$$

Hence

$$\begin{aligned} E[(\sup_{t \leq s \leq t+r} |B_t^H|)^m] &\leq \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty m y^{m-1} [\int_{r^{-H}y}^\infty e^{-x^2/2}\, dx]\, dy \\ &= \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-x^2/2} [\int_0^{r^H x} m y^{m-1}\, dy]\, dx \\ &= r^{Hm} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^m e^{-x^2/2}\, dx \end{aligned}$$

The result stated in Theorem 2.3 follows from the following relations:

$$\int_0^\infty x^m e^{-x^2/2}\, dx = (m-1)!! \text{ if } m \text{ is odd} \tag{2. 6}$$

and

$$\int_0^\infty x^m e^{-x^2/2}\, dx = \sqrt{\frac{\pi}{2}}\, (m-1)!! \text{ if } m \text{ is even} \tag{2. 7}$$

(cf. Gradshteyn and Ryzhik (2000)). For more details, see Muneya and Shieh (2009).

Debicki and Tomonek (2009) obtained bounds on the γ-th moment of B_T^{*H} for $\gamma > 0$ and $T > 0$. Let

$$K_T(H,\gamma) = E[(B_T^{*H})^\gamma] = E[\sup_{0\le t\le T} |B_t^H|^\gamma]. \tag{2. 8}$$

From the self-similarity of the fBm, it follows that

$$K_T(H,\gamma) = T^{\gamma H} K_1(H,\gamma). \tag{2. 9}$$

Theorem 2.4: (Debicki and Tomanek(2009)) Let $\gamma > 0$ and $H \in (0,1)$.

(i) If $0 < H < \frac{1}{2}$, then

$$K_T(H,\gamma) \ge T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\gamma/2} \Gamma(\frac{\gamma+1}{2}) \tag{2. 10}$$

and

(ii) if $\frac{1}{2} \le H < 1$, then

$$T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma}{2}} \Gamma(\frac{\gamma+1}{2}) \le K_T(H,\gamma) \le T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma}{2}+1} \Gamma(\frac{\gamma+1}{2}). \tag{2. 11}$$

Theorem 2.4 is a consequence of the following result, again due to Debicki and Tomanek (2009), for centered Gaussian processes.

Theorem 2.5: Let $X(0) = 0$ a.s, and $X \equiv \{X(t), t \ge 0\}$ be the centered Gaussian process with stationary increments and continuous strictly increasing variance function $Var(X(t)) = \sigma^2(t)$.

(i) If the function $\sigma^2(t)$ is sub-additive on $[0,T]$, in the sense that

$$\sigma^2(t) \le \sigma^2(s) + \sigma^2(t-s), 0 \le s \le t \le T,$$

then

$$E([\sup_{0\le t\le T} X(t)]^\gamma) \ge [\sigma^2(T)]^{\gamma/2} \frac{1}{\sqrt{\pi}} 2^{\gamma/2} \Gamma(\frac{\gamma+1}{2}) \tag{2. 12}$$

and

(ii) if the function $\sigma^2(t)$ is super-additive on $[0,T]$, in the sense that

$$\sigma^2(t) \ge \sigma^2(s) + \sigma^2(t-s), 0 \le s \le t \le T,$$

then

(2. 13) $$E([\sup_{0\le t\le T} X(t)]^{\gamma}) \le [\sigma^2(T)]^{\gamma/2}\frac{1}{\sqrt{\pi}}2^{\gamma/2}\Gamma(\frac{\gamma+1}{2}).$$

We include a modified version of the proof from Debicki and Tomanek (2009) for completeness.

Proof: (i) Suppose the function $\sigma^2(t)$ is concave on $[0,T]$. Let $\{W(t), t\ge 0\}$ be the standard Brownian motion. Define the stochastic process $Y(t) = W(\sigma^2(t)), t \ge 0$. Then the process $Y \equiv \{Y(t), t \ge 0\}$ is a centered Gaussian process with

$$Var(Y(t)) = Var(W(\sigma^2(t))) = \sigma^2(t) = Var(X(t)), 0 \le t \le T.$$

Furthermore, from the sub-additivity of the function $\sigma^2(.)$ on the interval $[0,T]$, it follows that

$$\begin{aligned} Var(Y(t) - Y(s)) &= Var(W(\sigma^2(t)) - W(\sigma^2(s))) \\ &= \sigma^2(t) - \sigma^2(s) \\ &\le \sigma^2(t-s) \text{ (by the sub-additivity of the function } \sigma^2(.)) \\ &= Var(X(t) - X(s)) \text{ (by stationary increments of process} X) \end{aligned}$$

for $0 \le s < t \le T$. Applying the Slepian's lemma (Theorem 1.2), it follows that

$$P(\sup_{0\le t\le T} X(t) > x) \ge P(\sup_{0\le t\le T} Y(t) > x)$$

Since

$$P(\sup_{0\le t\le T} Y(t) > x) = P(\sup_{0\le t\le \sigma^2(T)} W(t) > x),$$

it follows that

$$E([\sup_{0\le t\le T} X(t)]^{\gamma}) \ge E([\sup_{0\le t\le \sigma^2(T)} W(t)]^{\gamma}).$$

From the self-similarity of the Brownian motion, we get that

$$E([\sup_{0\le t\le \sigma^2(T)} W(t)]^{\gamma}) = [\sigma^2(t)]^{\gamma/2} E([\sup_{0\le t\le 1} W(t)]^{\gamma}).$$

Applying the reflection principle for the Wiener process $\{W(t), 0 \le t \le 1\}$, we get that

$$P(\sup_{0\le t\le 1} W(t) > x) = 2\ P(Z > x)$$

where Z is the standard Gaussian random variable. Hence

$$E([\sup_{0\leq t\leq 1} W(t)]^{\gamma}) = \int_0^{\infty} \gamma x^{\gamma-1} 2\phi(x)dx = \frac{1}{\sqrt{\pi}} 2^{\gamma/2}\Gamma(\frac{\gamma+1}{2}).$$

where $\phi(.)$ is the standard Gaussian probability density function. Combining the above inequalities, we get that

$$E([\sup_{0\leq t\leq T} X(t)]^{\gamma}) \geq [\sigma^2(T)]^{\gamma/2} \frac{1}{\sqrt{\pi}} 2^{\gamma/2}\Gamma(\frac{\gamma+1}{2}).$$

Similar arguments can be used to prove (ii) in Theorem 2.5.$\diamondsuit$

Debicki and Tomanek (2009) has assumed that the function $\sigma^2(t)$ is concave in (i) and convex in (ii) of Theorem 2.5. We could not justify the arguments under these assumptions and hence we replaced "concavity"in (i) by "sub-additivity" in (i) and "convexity" in (ii) by "super-additivity" of the function $\sigma^2(t)$.

Proof of Theorem 2.4: In view of (2.9), it is sufficient to prove the results in Theorem 2.4 for the case $T = 1$. Observe that

$$\begin{aligned} P[\sup_{0\leq t\leq 1} B_t^H > x] &\leq P[\sup_{0\leq t\leq 1} |B_t^H| > x] \\ &\leq 2\, P[\sup_{0\leq t\leq 1} B_t^H > x]. \end{aligned}$$

An application of this inequality combined with results in Theorem 2.5 proves the lower bound in (i) in Theorem 2.4 and the upper bound in (ii) of Theorem 2.4. The lower bound in (ii) of Theorem 2.4 can be obtained by observing that

$$\begin{aligned} E([\sup_{0\leq t\leq 1} |B_T^H|]^{\gamma}) &\geq E[|B_1^H|^{\gamma}] \\ &= E[|Z|^{\gamma}] \\ &= \frac{1}{\sqrt{\pi}} 2^{\gamma/2}\Gamma(\frac{\gamma+1}{2}) \end{aligned}$$

where Z is the standard Gaussian random variable.$\diamondsuit$.

The next two results give bounds on the expectations of the exponential of the supremum for the fBm over a finite interval $[0, T]$.

Theorem 2.6: Suppose the process B^H is the standard fractional Brownian motion with Hurst index $H \in [\frac{1}{2}, 1)$. Then, for any $\lambda > 0$,

$$E[\exp\{\lambda \sup_{0\leq t\leq T} |B_t^H|\}] \leq 1 + \lambda\sqrt{8\pi}T^H e^{\frac{\lambda^2 T^{2H}}{2}}. \tag{2. 14}$$

Theorem 2.7: Suppose B^H is the standard fractional Brownian motion with Hurst index $H \in [\frac{1}{2}, 1)$. Then, for any $\lambda > 0, T > 0$ such that $\lambda T^{2H} < \frac{1}{2}$,

$$E[\exp\{\lambda \sup_{0\leq t\leq T} (B_t^H)^2\}] \leq 1 + 2\lambda + 8\lambda \frac{T^{2H}}{\sqrt{1 - 2\lambda T^{2H}}}. \tag{2. 15}$$

Let F be the distribution function of the random variable B_T^{*H}. Note that, for any $\lambda > 0$,

$$\begin{aligned} E[\exp(B_T^{*H})] &= \int_0^\infty e^{\lambda x}\, dF(x) \\ &= -\int_0^\infty e^{\lambda x}\, d(1 - F(x)) \\ &= 1 + \lambda \int_0^\infty e^{\lambda x}(1 - F(x))\, dx \\ &= 1 + \lambda \int_0^\infty e^{\lambda x} P(B_T^{*H} > x)\, dx \end{aligned}$$

Now we use the bounds on $P(B_T^{*H} > x)$ obtained earlier to get the inequality in Theorem 2.6. For details, see Mishra and Prakasa Rao (2013). Similarly we note that

$$\begin{aligned} E(\exp[(B_T^{*H})]^2) &= \int_0^\infty e^{\lambda x^2}\, dF(x) \\ &= 1 + 2\lambda \int_0^\infty x e^{\lambda x^2}(1 - F(x))\, dx \\ &= 1 + 2\lambda \int_0^\infty x e^{\lambda x} P(B_T^{*H} > x)\, dx \end{aligned}$$

to obtain the inequality in Theorem 2.7. For details, see Prakasa Rao (2013).

3 Maximal inequalities and first passage times

Let the process $B^H = \{B_t^H, t \geq 0\}$ be the standard fractional Brownian motion with Hurst index H. For any $a > 0$, define

$$\eta_a^H = \inf\{t \geq 0 : |B_t^H| \geq a\}$$

and

$$\zeta_a^H = \inf\{t \geq 0 : B_t^H \geq a\}.$$

Let

$$B^{*H}_T = \sup_{0 \le t \le T} |B^H_t|$$

and

$$S^H_T = \sup_{0 \le t \le T} B^H_t.$$

Note that the random variables B^{*H}_T and S^H_T are non-negative since $B^H_0 = 0$ a.s.

Theorem 3.1: (Vardar (2011)) For any $H \in (\frac{1}{2}, 1)$ and for any $T > 0$,

$$(B^{*H}_T)^2 \triangleq (\frac{T}{\eta^H_1})^{2H}$$

and

$$(S^H_T)^2 \triangleq (\frac{T}{\zeta^H_1})^{2H}.$$

Furthermore

$$E[(S^H_T)^2] \le T^{2H}.$$

Proof : From the self-similarity of the fBm B^H, it follows that

$$\{B^H_{at}, t \ge 0\} \triangleq \{a^H B^H_t, t \ge 0\}$$

for any $a > 0$. Hence, for any $x > 0$,

$$\begin{aligned} P[(\frac{a}{\eta^H_1})^{2H} \le x] &= P[\eta^H_1 \ge \frac{a}{x^{1/2H}}] \\ &= P[\sup_{0 \le t \le \frac{a}{x^{1/2H}}} |B^H_t| \le 1] \\ &= P[\sup_{0 \le t \le a} |B^H_{\frac{t}{x^{1/2H}}}| \le 1] \\ &= P[\sup_{0 \le t \le a} |B^H_u| \le \sqrt{x}] \ \ \text{(by self-similarity of the process } B^H) \\ &= P[(B^{*H}_a)^2 \le x)]. \end{aligned}$$

Hence

$$(B^{*H}_a)^2 \triangleq (\frac{a}{\eta_1})^{2H}.$$

Similar arguments show that

$$(S^H_a)^2 \triangleq (\frac{a}{\zeta_1})^{2H}.$$

Furthermore, for any $x > 0$,

$$\begin{aligned} P(\sup_{0\le u\le a}|B_u^H| \le \sqrt{x}) &= P(\sup_{0\le u\le a} B_u^H \le \sqrt{x} \text{ and } \inf_{0\le u\le a} B_u^H \ge -\sqrt{x}) \\ &\le P(\sup_{0\le u\le a} B_u^H \le \sqrt{x}) \\ &= P((S_a^H)^2 \le x). \end{aligned}$$

Therefore, for any $x > 0$,

$$P((S_a^H)^2 \le x) = P((\frac{a}{\zeta_1})^{2H} \le x)$$

and hence

$$\begin{aligned} E[(S_a^H)^2] &= a^{2H} E[(\frac{1}{\zeta_1})^{2H}] \\ &= a^{2H}\int_0^\infty E(e^{-x\zeta_1^{2H}})\,dx \end{aligned} \tag{3. 1}$$

from the elementary observation that for any positive random variable Z,

$$E(\frac{1}{Z}) = \int_0^\infty E(e^{-xZ})dx.$$

It is known that, for any $H > \frac{1}{2}$,

$$E[e^{-\lambda\zeta_a^{2H}}] \le e^{-a\sqrt{2\lambda}}$$

for every $\lambda > 0$ and $a > 0$ (cf. Decreusefond and Nualart (2008)). Applying this inequality in (3.1), we get that

$$E[({B^*}_a^H)^2] \le a^{2H}\int_0^\infty e^{-\sqrt{2x}}dx = a^{2H}.$$

♢

As a corollary to Theorem 3.1, it follows that $E({B^*}_a^H) \le a^H$ and $E(S_a^H) \le a^H$ by the elementary inequality $[E(Z)]^2 \le E[Z^2]$. Furthermore,

$$P({B^*}_a^H > x) \le \frac{a^H}{x}$$

and

$$P(S_a^H > x) \le \frac{a^H}{x}$$

for any $a > 0$ and $x > 0$ by the Chebyshev's inequality.

Results discussed above are due to Vardar (2011). An improved upper bound for $P(S_a^H > x)$ was also obtained in Vardar (2011). It was shown that

$$P(S_a^H > x) \leq \frac{\sqrt{2}a^H}{x\sqrt{\pi}}, a > 0, x > 0.$$

and hence, for any $a > 0$,

$$E(S_a^H) \leq \frac{\sqrt{2}}{\sqrt{\pi}} a^H. \tag{3. 2}$$

Michna (1999) studied the asymptotic behaviour of the tail probabilities and first passage times for fractional Brownian motion B^H, $H \in (\frac{1}{2}, 1)$ with linear drift.

4 Maximal Inequalities Leading to Lower Bounds

We now discuss some maximal inequalities leading to lower bounds for functionals of the fBm. An important inequality which was found to be useful in such discussions is the Khatri-Sidak inequality proved in Khatri (1967) and Sidak (1967, 1968). Jogdeo (1970) and Schechtman et al. (1998) gave simplified proofs of the result.

Theorem 4.1: (Khatri (1967); Sidak (1967, 1968)) If $(X_1, \ldots, X_n)$ is a centered Gaussian random vector, then

$$P(\max_{1 \leq i \leq n} |X_i| \leq x) \geq P(|X_1| \leq x)\ P(\max_{2 \leq i \leq n} |X_i| \leq x) \tag{4. 1}$$

for any $x > 0$.

Repeated applications of Theorem 4.1 show that

$$P(\max_{1 \leq i \leq n} |X_i| \leq x) \geq \Pi_{i=1}^n P(|X_i| \leq x) \tag{4. 2}$$

for any centered Gaussian random vector $(X_1, \ldots, X_n)$. Shao (2003) showed that

$$P(\max_{1 \leq i \leq n} |X_i| \leq 1) \geq 2^{-\min(k,n-k)} P(\max_{1 \leq i \leq k} |X_i| \leq 1)\ P(\max_{k+1 \leq i \leq n} |X_i| \leq 1) \tag{4. 3}$$

for any $1 \leq k \leq n$. Li (1999) proved the following general result.

Theorem 4.2: Let μ be a centered Gaussian probability measure on a separable Banach space E. Then, for any $0 < \lambda < 1$, and any symmetric convex sets A and B contained in E,

$$P(X \in A, Y \in B) \geq P(X \in \lambda A)\ P(Y \in \sqrt{1 - \lambda^2} B) \tag{4. 4}$$

for any centered jointly Gaussian vectors X and Y in E.

Suppose $\{X_t, t \in T\}$ is a centered Gaussian process. As a consequence of the Khatri-Sidak inequality, it follows that

$$P(\sup_{t\in A}|X_t| \le x, |X_{t_0}| \le x) \ge P(|X_{t_0}| \le x)\ P(\sup_{t\in A}|X_t| \le x) \tag{4. 5}$$

for every subset $A \subset T, t_0 \in T$ and $x > 0$. If there is a countable set J and a Gaussian process Y indexed by the set J such that

$$[\sup_{t\in J}|Y_t| \le x] \Rightarrow [\sup_{t\in T}|X_t| \le x],$$

then

$$P(\sup_{t\in T}|X_t| \le x) \ge \Pi_{t\in J}P(|Y_t| \le x).$$

Li and Shao (2001) discuss lower bounds for small ball probabilities for general Gaussian processes extensively. We discuss special cases dealing with the fBm in some detail. Shao (1999) obtained the following correlation inequality for a fBm B^H : there exists a constant $d_H > 0$ such that,

$$\begin{aligned} &P(\sup_{0\le t\le a}|B_t^H| \le x, \sup_{0\le t\le b}|B_t^H - B_a^H| \le y) \\ \ge\ & d_H P(\sup_{0\le t\le a}|B_t^H| \le x)\ P(\sup_{a\le t\le b}|B_t^H - B_a^H| \le y) \end{aligned}$$

for any $0 < a < b, x > 0, y > 0$.

The following result due to Csorgo and Shao (1994) and Kuelbs et al. (1995) gives a lower bound for centered Gaussian processes.

Theorem 4.3: Let the process $\{X(t), 0 \le t \le 1\}$ be a centered Gaussian process with $X(0) = 0$. Suppose there exists a function $\sigma^2(h)$ such that

$$E|X(s) - X(t)|^2 \le \sigma^2(|t - s|)$$

and there exists constants c_1, c_2 such that $0 < c_1 \le c_2 < 1$ and

$$c_1\ \sigma(\min(2h, 1)) \le \sigma(h) \le c_2\ \sigma(\min)2h, 1)$$

for $0 \leq h \leq 1$. then , there exists a positive constant K depending on c_1 and c_2 such that

$$P(\sup_{0 \leq t \leq 1} |X(t| \leq \sigma(\epsilon)) \geq e^{-\frac{K}{\epsilon}}. \tag{4. 6}$$

The following example, due to Lifshits (1999), indicates the methods involved in obtaining upper and lower bounds for small ball probabilities.

Let $H > 0$ and $\{\psi_i, i \geq 0\}$ be independent standard Gaussian random variables. Consider the function $f(t) = 1 - |2t - 1|, 0 \leq t \leq 1$ and let $\{u\}$ denote the fractional part of any real number u. Let

$$X(t) = \psi_0 t + \sum_{i=1}^{\infty} 2^{-iH} \psi_i f(\{2^i t\}), 0 \leq t \leq 1. \tag{4. 7}$$

It can be seen that the process $\{X(t), 0 \leq t \leq 1\}$ is a centered Gaussian process with $E|X(t) - X(s)|^2 \geq c|t - s|^{2H}, 0 \leq s, t \leq 1$ for some constant $c > 0$. . Note that

$$\begin{aligned}
P(\sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon) &\geq P(\sum_{i01}^{\infty} 2^{-iH} |\psi_i| \leq \epsilon) \\
&\geq P(|\psi_i| \leq \epsilon 2^{iH/2}(1 - 2^{H/2}), i \geq 0) \\
&= \Pi_{i=0}^{\infty} P(|\psi_i| \leq \epsilon 2^{iH/2}(1 - 2^{H/2})) \\
&\geq \exp(-k_1 \log^2(\frac{1}{\epsilon})))
\end{aligned}$$

for some positive constant $k_1 > 0$. On the other hand,

$$\begin{aligned}
P(\sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon) &\leq P(\sup_{k \geq 2} |X(2^{-k})| \leq \epsilon) \\
&\leq P(\sup_{k \geq 2} |\sum_{i=1}^{k-1} 2^{-iH} \psi_i 2^{-(k-i-1)}| \leq \epsilon) \\
&\leq P(\sup_{k \geq 2} |\psi_k| \leq 2\epsilon 2^{kH}) \\
&= \Pi_{k=2}^{\infty} P(|\psi_0 \leq 2\epsilon 2^{kH}) \\
&\leq \exp(-k_2 \log^2(\frac{1}{\epsilon})))
\end{aligned}$$

for some positive constant $k_2 > 0$.

The following result is an application of Theorem 4.3 (cf. Monrad and Rootzen (1995); Shao (1993)).

Theorem 4.4: Let the process B^H be the fBm with index $H \in (0,1)$. Then there exists constants $0 < K_1 \leq K_2 < \infty$ depending only on H such that, for every $0 < \epsilon \leq 1$,

$$-K_2 \epsilon^{-1/H} \leq \log P(\sup_{0 \leq t \leq 1} |B_t^H| \leq \epsilon) \leq -K_1 \epsilon^{-1/H}. \tag{4. 8}$$

Similar results for some functionals of the fBm were obtained in Kuelbs et al. (1995) and Li and Shao (1999, 2001).

Let

$$Y_t^H = \int_0^t B_s^H ds, 0 \leq t \leq 1.$$

Li and Shao (2001) proved that there exists constants $0 < K_1 \leq K_2 < \infty$ depending only on H such that, for every $0 < \epsilon \leq 1$,

$$-K_2 \epsilon^{-1/(1+H)} \leq \log P(\sup_{0 \leq t \leq 1} |Y_t^H| \leq \epsilon) \leq -K_1 \epsilon^{-1/(1+H)}. \tag{4. 9}$$

The following result due to Li and Linde (1998) and Shao (2003) gives the exact rate of convergence for small ball probabilities for fractional Brownian motion.

Theorem 4.5: Let $B^H = \{B_t^H, t \geq 0\}$ be the fBm with Hurst index H. Let

$$W_H(t) = \int_0^t (t-s)^{(2H-1)/2} dW(s)$$

where $\{W(t), t \geq 0\}$ is the standard Wiener process. Then

$$\begin{aligned} &\lim_{\epsilon \to 0} \epsilon^{1/H} \log\ P(\sup_{0 \leq t \leq 1} |B_t^H| \leq \epsilon) \\ = &\lim_{\epsilon \to 0} \epsilon^{1/H} \log\ P(\sup_{0 \leq t \leq 1} |W_H(t)| \leq \sqrt{a_H}\epsilon) \\ = &-c_H \end{aligned} \tag{4. 10}$$

where $0 < c_H < \infty$,

$$a_H = \frac{1}{2H} + \int_{-\infty}^0 ((1-s)^{(2H-1)/2} - (-s)^{(2H-1)/2})^2\, ds$$

and

$$(\frac{0.08}{\sqrt{2H}})^{1/H} < c_H < (\frac{10}{\sqrt{2H}})^{1/H}$$

for $0 < H < \frac{1}{2}$.

Kuelbs and Li (2000) obtained the following related result.

Theorem 4.6: Let $B^H = \{B_t^H, t \geq 0\}$ be a fBm with Hurst index $H \in (0,1)$ Let $\rho(.)$ be a non-negative bounded function on $[0,1]$ such that $[\rho(t)]^{1/H}$ is Riemann integrable on $[0,1]$. Then

$$\lim_{\epsilon \to 0} \epsilon^2 \log\ P(\sup_{0 \leq t \leq 1} |\rho(t) B_t^H| \leq \epsilon) = -c_H \int_0^1 [\rho(t)]^{1/H} dt \tag{4. 11}$$

where c_H is as defined in Theorem 4.5.

Talagrand (1996) obtained the following integral test for the fBm.

Theorem 4.7: Let $B^H = \{B_t^H, t \geq 0\}$ be a fBm with Hurst index $H \in (0,1)$. Let $a(.)$ be a non-decreasing function such that the function $\frac{a(t)}{t^H}$ is bounded over the interval $(0, \infty)$. Then

$$P(\sup_{0 \leq s \leq t} |B_s^H| < a(t) \text{ infinitely often})$$

is zero or one according as

$$\int_0^\infty [a(t)]^{-1/H} \psi(a(t) t^{-H})\ dt$$

is convergent or divergent. Here $\psi(h) = P(\sup_{0 \leq s \leq 1} |B_s^H| \leq h)$.

Kuelbs and Li (2000) proved a Wichura type functional law of iterated logarithm for a fBm.

Let $B^H = \{B_t^H, t \geq 0\}$ be the fBm with Hurst index $H \in (0,1)$. Let

$$B^{*H}_T = \sup_{0 \leq s \leq t} |B_s^H|$$

and

$$H_n(t) = \frac{M_{nt}^H}{(c_H n^{2H} / \log\log n)^{1/2}}$$

where c_H is as given in Theorem 4.5. Let $\mathcal{M}$ be the class of non-decreasing functions $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and which are right continuous except possibly at zero. Let

$$K_\alpha = \{f \in \mathcal{M} : \int_0^\infty [f(t)]^{-1/H} dt \leq 1\}.$$

Let the set $\mathcal{M}$ be equipped with the topology of weak convergence, that is, pointwise convergence at all the continuity points of the limit. Kuelbs and Li (2000) proved the following Wichura type functional law of the iterated logarithm.

Theorem 4.8: (Kuelbs and Li (2000)) The sequence $\{H_n(t)\}$ is relatively compact and the set of all possible subsequential limits of the sequence $\{H_n(t)\}$ in the weak topology is the set K_α.

Related results on increments of the fBm are discussed in Li and Shao (2001) in a survey on inequalities for Gaussian processes.

5 Limit theorems for Maximal Inequalities for fBM with Polynomial Drift

We will now discuss some limit theorems for some maximal inequalities for a fBm with Hurst index $H \in (\frac{1}{2}, 1)$ with polynomial drift. Results discussed here are from Prakasa Rao (2013).

Theorem 5.1: Suppose $a > 0, \delta > 0$. Then, for any $k \geq 1$,

$$(5.\ 1)\quad -\frac{a^2}{2}(\frac{k}{H})^{2k/(2k-2H)}\frac{H}{k-H} \leq \limsup_{T\to\infty} \frac{\log\ P(\sup_{0\leq t\leq T}[B_t^H + at^k] \leq \delta a)}{T^{2k-2H}} \leq -\frac{a^2}{2}.$$

For $H = \frac{1}{2}$, the process B^H is the Brownian motion and the result in Theorem 5.1 generalizes Theorem 2.2 in Li (2010) for the case of the Brownian motion.

Let $g(t) = a_k t^k + a_{k-1}t^{k-1} + \ldots + a_1 t$ be a polynomial of degree k with $a_k > 0$. As a corollary to Theorem 5.1, we can obtain the following result.

Theorem 5.2: Suppose $x > 0$. Then, for any $k \geq 1$,

$$(5.\ 2)\quad -\frac{a^2}{2}(\frac{k}{H})^{2k/(2k-2H)}\frac{H}{k-H} \leq \limsup_{T\to\infty} \frac{\log\ P(\sup_{0\leq t\leq T}[B_t^H + g(t)] \leq x)}{T^{2k-2H}} \leq -\frac{a^2}{2}.$$

Applying the result in Theorem 5.2 for the process $-B^H$ which is again a fBm with Hurst index H, the following result follows.

Theorem 5.3: Suppose $x > 0$. Then, for any $k \geq 1$,

$$(5.\ 3)\quad -\frac{a^2}{2}(\frac{k}{H})^{2k/(2k-2H)}\frac{H}{k-H} \leq \limsup_{T\to\infty} \frac{\log\ P(\inf_{0\leq t\leq T}[B_t^H - g(t)] \geq -x)}{T^{2k-2H}} \leq -\frac{a^2}{2}.$$

We now discuss some other results dealing with the supremum of a fBm B^H on the interval $[0, T]$.

Theorem 5.4: Let $a > 0, \sigma > 0$ and $T > 0$. Then

$$(5.\ 4)\qquad \limsup_{\delta\to\infty} \frac{\log\ P(\sup_{0\le t\le T}(\sigma B_t^H + at) \ge \delta a)}{\delta^2} \le -\frac{a^2}{2\sigma^2 T}.$$

Duffield and O'Connell (1995) proved that,

$$(5.\ 5)\qquad \lim_{\delta\to\infty} \delta^{-2(1-H)} \log\ P(\sup_{t\ge 0}(B_t^H - at) > \delta) = -\inf_{c>0} c^{-2(1-H)} \frac{(c+a)^2}{2}.$$

This result deals with the tail probability of the supremum with linear drift over an infinite horizon. Debicki et al. (1998) proved that, for $H \in [\frac{1}{2}, 1)$,

$$(5.\ 6)\qquad \lim_{\delta\to\infty} \delta^{-2(1-H)} \log\ P(\sup_{t\ge 0}(B_t^H - at) > \delta) = -\frac{1}{2}(\frac{a}{H})^{2H}(\frac{1}{1-H})^{2-2H}.$$

They have also studied the asymptotic behaviour of the tail of the distribution function

$$P(\sup_{t\ge 0}(Z(t) - ct) > x)$$

as $x \to \infty$ when the process $\{Z(t), t \ge 0\}$ is a fractional Brownian motion or a nonlinearly scaled Brownian motion or a specific integrated stationary Gaussian process. This result generalizes the earlier work of Norros (1994) and Duffield and O'Connell (1995). Their results deal with processes with linear drift. Michna (1998,1999) studied the asymptotic behaviour of tail probabilities of the supremum of a fractional Brownian motion with linear drift over a finite time interval.

Theorem 5.5: Let $a < 0, \sigma > 0$ and $T > 0$. Then

$$(5.\ 7)\qquad \liminf_{\delta\downarrow 0} \frac{1}{-\log\delta} \log\ P(\sup_{0\le t\le T}(\sigma B_t^H + at) \le -\delta a) \ge -1.$$

Theorem 5.6: For any $k \ge 2, a > 0$ and $T > 0$,

$$(5.\ 8)\qquad \lim_{\delta\to\infty} \frac{\log\ P(\sup_{0\le t\le T}(\sigma B_t^H + at^k) \ge \delta a)}{\delta^2} = -\frac{a^2}{2T^{2H}}.$$

The next result extends the results obtained above for polynomial drift.

Theorem 5.7: Let $g(t) = a_k t^k + a_{k-1} t^{k-1} + \ldots + a_1 t$ with $a_k > 0$. Then, for any $T > 0$,

$$\lim_{x\to\infty} \frac{\log\ P(\sup_{0\leq t\leq T}(B_t^H + g(t)) \geq x)}{x^2} = -\frac{1}{2T^{2H}}. \tag{5. 9}$$

6 Limit Theorems for Maximal Inequalities for Increments of Fractional Brownian Motion

Let the process B^H be the fractional Brownian motion with Hurst index $H \in (0,1)$ with $B_0^H = 0$ a.s. From the self-similarity of the fBm B^H and from the fact that the increments of the process B^H are stationary, it follows that, for any $r > 0$,

$$\begin{aligned}
\sup_{s\leq t\leq s+r} |B_t^H - B_s^H| &\triangleq \sup_{0\leq t\leq r} |B_t^H - B_0^H| \\
&= \sup_{0\leq t\leq r} |B_t^H| \ \text{ a.s} \\
&\triangleq r^H \sup_{0\leq t\leq 1} |B_t^H|
\end{aligned}$$

and hence, for any $x > 0$,

$$P(\sup_{s\leq t\leq s+r} |B_t^H - B_s^H| \geq x) = P(\sup_{0\leq t\leq 1} |B_t^H| > r^{-H} x). \tag{6. 1}$$

In view of the above equation, it is possible to obtain maximal inequalities for the increments of the process fBm from the results discussed in earlier sections. However there are some other results which need additional arguments. We will now discuss such results. Let

$$I_t^H = \inf_{0\leq s\leq t} B_s^H,$$

$$S_t^H = \sup_{0\leq s\leq t} B_s^H,$$

$$R_t^H = S_t^H - I_t^H,$$

and

$$\nu_t^H = \sup_{0\leq u\leq v\leq t} (B_u^H - B_v^H) = \sup_{0\leq v\leq t} (\sup_{0\leq u\leq v} (B_u^H - B_v^H)).$$

Let

$$Y_a^H = S_a^H - B_a^H$$

for any $a > 0$. Vardar (2011) proved that, for any $y > 0$,

$$P(Y_a^H \leq y) \geq 1 - \frac{\sqrt{2}a^H}{y\sqrt{\pi}}. \tag{6. 2}$$

Caglar and Vardar (2012) proved the following theorem.

Theorem 6.1: Let B^H be the fractional Brownian motion with $B_0^H = 0$ a.s. and $H \in (\frac{1}{2}, 1)$. Then, for any $a > 0, y > 0$,

$$\frac{\sqrt{2}a^H}{2\sqrt{\pi}} \leq E(\nu_a^H) \leq \frac{2\sqrt{2}a^H}{\sqrt{\pi}} \tag{6. 3}$$

and

$$P(\nu_a^H > y) < P(R_a^H \geq y) \leq \frac{2\sqrt{2}a^H}{y\sqrt{\pi}}. \tag{6. 4}$$

Let

$$X_v^H = \sup_{0 \leq u \leq v} (B_u^H - B_v^H), v \geq 0.$$

The process X^H is called the loss process. It is self-similar and the random variable X_v^h has the same distribution as the random variable S_v^H. This observation follows from the self-similarity of the fBm and the fact that the fBm has stationary increments.

Theorem 6.2: For any $x > 0$,

$$P(\nu_t > x) \geq \bar{\Phi}(\frac{x}{t^H})$$

where $\bar{\Phi}(x) = 1 - \Phi(x), -\infty < x < \infty$ and $\Phi(.)$ is the standard normal distribution function.

The result stated above can be proved by the following arguments.

Proof of Theorem 6.2: Note that, for $0 \leq v \leq t$,

$$\begin{aligned} P(X_v^H > x) &= P(\sup_{0 \leq u \leq v} B_u^H > x) \\ &\geq \sup_{0 \leq u \leq v} P(B_u^H > x) \\ &= \sup_{0 \leq u \leq v} \bar{\Phi}(\frac{x}{u^H}) \\ &= \bar{\Phi}(\frac{x}{v^H}). \end{aligned}$$

Hence

$$
\begin{aligned}
P(\nu_t^H > x) &= P(\sup_{0\leq v\leq t} X_v^h > x) \\
&\geq \sup_{0\leq v\leq t} P(X_v^h > x) \\
&\geq \sup_{0\leq v\leq t} \bar{\Phi}(\frac{x}{v^H}) \\
&= \bar{\Phi}(\frac{x}{t^H}).
\end{aligned}
$$

As an application of Theorem 6.2, we see that

$$
\liminf_{x\to\infty} \frac{1}{x^2} \log\ P(\nu_t^H \geq x) \geq \lim_{x\to\infty} \frac{1}{x^2} \log\ \bar{\Phi}(\frac{x}{t^H}) = -\frac{1}{2t^{2H}}. \tag{6. 5}
$$

Using the second inequality in Theorem 1.4, it can be shown that, for $x > \eta = E[\sup_{0\leq u\leq v\leq t}(B_u^H - B_v^H)]$,

$$
\begin{aligned}
P(\nu_t^H > x) &= P(\sup_{0\leq u\leq v\leq t}(B_u^H - B_v^H) > x) \\
&\leq 2e^{-\frac{1}{2}\frac{(x-\eta)^2}{t^{2H}}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\limsup_{x\to\infty} \frac{1}{x^2} \log\ P(\nu_t^H \geq x) &\leq \lim_{x\to\infty}[\frac{\log 2}{x^2} - \frac{1}{x^2}\frac{(x-\eta)^2}{2t^{2H}}] \\
&= -\frac{1}{t^{2H}}.
\end{aligned} \tag{6. 6}
$$

Combining the above results, we have the following result due to Caglar and Vardar (2012).

Theorem 6.3: Let B^H be the fractional Brownian motion with index $H \in (\frac{1}{2}, 1)$. Let

$$
\nu_t^H = \sup_{0\leq u\leq v\leq t}(B_u^H - B_v^H).
$$

Then, for any $t > 0$,

$$
\lim_{x\to\infty} \frac{1}{x^2} \log\ P(\nu_t^H > x) = -\frac{1}{2t^{2H}}. \tag{6. 7}
$$

Let $Y_t^H = \mu_t + \sigma B_t^H, t \geq 0$ where $-\infty < \mu < \infty$ and $\sigma > 0$. The process Y^H is the fractional Brownian motion with linear drift. Let

$$\gamma_t^H = \sup_{0 \leq u \leq v \leq t} (Y_u^H - Y_v^H).$$

Caglar and Vadar (2012) proved that, for any $t > 0$,

$$\lim_{x \to \infty} \frac{1}{x^2} \log\ P(\gamma_t^H > x) = -\frac{1}{2\sigma^2 t^{2H}}. \tag{6. 8}$$

The maximum loss of a stochastic process $\{X_s, 0 \leq s \leq t\}$ over an interval $[0, t]$, is defined to be

$$\sup_{0 \leq u \leq v \leq t} (X_u - X_v).$$

Caglar and Vardar-Acar (2013) obtained bounds and asymptotic results on the distribution of maximum loss of the the standard fBm B^H with drift when $H \in (1/2, 1)$. . Let

$$Y_t^H = \mu t + \sigma B_t^H, t \geq 0,$$

$$I_t^{H,\mu} = \inf_{0 \leq v \leq t} Y_v^H, t \geq 0,$$

$$S_t^{H,\mu} = \sup_{0 \leq v \leq t} Y_v^H, t \geq 0,$$

$$R_t^{H,\mu} = S_t^{H,\mu} - I_t^{H,\mu}, t \geq 0,$$

$$L_t^{H,\mu} = \sup_{0 \leq u \leq t} (Y_u^H - Y_t^H), t \geq 0,$$

and

$$M_t^H = \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v), t \geq 0,$$

Since the fBm is self-similar and has stationary increments, it can be shown that the process $L^{H,0}$ is self-similar and the random variables $L_v^{H,0}$ and $S_v^{H,0}$ have the same distribution for every $v \geq 0$. The following result is due to Caglar and Vardar-Acar (2013).

Theorem 6.4: For the fBm with Hurst index $H \in (1/2, 1)$, drift $\mu \in R$, and $\sigma > 0$,

$$\frac{\sqrt{2}\sigma t^H}{2\sqrt{\pi}} + \min(\mu, 0)t \leq E(M_t^{H,\mu}) \leq E(R_t^{H,\mu}) \leq \frac{2\sqrt{2}\sigma t^H}{\sqrt{\pi}} + |\mu|t$$

and

$$1 - \Phi((x+\mu t)/(\sigma t^H)) \leq P(M_t^{H,\mu} > x) \leq P(R_t^{H,\mu} \geq x) \leq \frac{2\sqrt{2}\sigma t^H}{x\sqrt{\pi}} + \frac{|\mu| t}{x}$$

for $x > 0$ and $t > 0$.

Kuelbs et al. (1995) obtained the following result dealing with increments of the fBm.

Theorem 6.5: (Kuelbs et al. (1995)) Let B^H be the fractional Brownian motion with Hurst index $H \in (0,1)$. Let $0 \leq \beta < H$. Then there exists $0 < K_1 \leq K_2 < \infty$ depending only on H and β such that, for all $0 \leq \epsilon \leq 1$,

$$(6.\ 9) \qquad -K_2 \epsilon^{-1/(H-\beta)} \leq \log\ P(\sup_{0 \leq s,t \leq 1} \frac{|B_s^H - B_t^H|}{|s-t|^\beta} \leq \epsilon) \leq -K_1 \epsilon^{-1/(H-\beta)}.$$

Li and Shao (1999) proved a related result dealing with integrals of increments of a fBm.

Theorem 6.5: (Li and Shao (1999)) Let B^H be a fractional Brownian motion with Hurst index $H \in (0,1)$. Let $p > 0, 0 \leq q < 1 + pH, q \neq 1$. Then there exists $0 < K_1 \leq K_2 < \infty$ depending only on p, q and H such that, for all $0 \leq \epsilon \leq 1$,

$$(6.\ 10) \qquad -K_2 \epsilon^{-\theta} \leq \log\ P(\int_0^1 \int_0^1 \frac{|B_s^H - B_t^H|^p}{|s-t|^q}\, dt ds \leq \epsilon) \leq -K_1 \epsilon^{-\theta}$$

where $\theta = \frac{1}{H - \max(0, q-1)}$.

7 One-sided Exit Problem for fractional Brownian Motion

We now consider the one-sided exit problem for the fractional Brownian motion. This deals with the study of lower tail probabilities of the supremum of the fractional Brownian motion over finite time interval (cf. Li and Shao (1964)).

Let $B^H = \{B_t^H, t \geq 0\}$ be the fractional Brownian motion with index $H \in (0,1)$. Let

$$S_T^H = \sup_{0 \leq t \leq T} B_t^H$$

and

$$\zeta_a^H = \inf\{t \geq 0 : B_t^H \geq a\}, a > 0.$$

From the self-similarity of the fBm B^H, it follows that

$$P(S_T^H \leq x) = P(S_1^H \leq \frac{x}{T^H}).$$

Hence the study of asymptotic limit of $P(S_T^H \leq x)$ as $T \to \infty$ for fixed x is equivalent to that as $x \to 0$ for fixed T. It is obvious that

$$P(S_T^H < a) = P(\zeta_a^H > T).$$

Sinai (1997) showed that the distribution of the random variable ζ_a^H has a probability density function $p(.)$ and, for $H > \frac{1}{2}$,

$$p(T) \leq \text{ Constant } (2H-1)T^{H-\frac{1}{2}+b(H)}$$

for T large where $|b(H)| \leq$ Constant $(2H-1)$. Molchan (1999, 2000) proved that

$$T^{-(1-H)}e^{-k\sqrt{\log T}} \leq P(\sup_{0\leq t\leq T} B_t^H \leq 1) \leq T^{-(1-H)}e^{k\sqrt{\log T}}$$

for some positive constant k and T large. In a recent work, Aurzada (2011) proved the following result.

Theorem 7.1: Let $\{B_t^H, t \geq 0\}$ be the fBm with index $H \in (0,1)$. Then there is a positive constant C_H depending on h such that, for large T,

$$T^{-(1-H)}(\log T)^{-C_H} \leq P(\sup_{0\leq t\leq T} B_t^H \leq 1) \leq T^{-(1-H)}(\log T)^{C_H}. \tag{7. 1}$$

It can be shown that the constant C_H in the lower bound is greater than $\frac{1}{2H}$ and the constant C_H in the upper bound is greater than $\frac{2}{H} - 1$. Since the process B^H is self-similar, the result in Theorem 7.1 can be generalized to the lower tail of the supremum of the fBm.

Theorem 7.2: Let $\{B_t^H, t \geq 0\}$ be the fBm with index $H \in (0,1)$. Then there is a positive constant C_H depending on h such that, for small $\epsilon > 0$,

$$\epsilon^{\frac{(1-H)}{H}}|\log \epsilon|^{-C_H} \leq P(\sup_{0\leq t\leq 1} B_t^H \leq \epsilon) \leq \epsilon^{\frac{(1-H)}{H}}|\log \epsilon|^{C_H} \tag{7. 2}$$

Li and Shao (2004) proved that

$$-\lim_{T\to\infty} \frac{1}{T} \log P(\sup_{0\leq t\leq T} e^{-tH} B_{e^t}^H \leq 0) = d_H \tag{7. 3}$$

exists, and , as $x \to 0$,

$$P(\sup_{0\leq t\leq 1} B_t^H \leq x) = x^{2d_H/2H+o(1)} \tag{7. 4}$$

Results in Molchan (1999, 2000) show that $d_H = 1 - H$. Unilateral small deviations for some processes related to the fBm are discussed in Molchan (2008). Li and Shao (2004) obtained the following result on the lower tail probability for the supremum of the fBm.

Theorem 7.3: Let B^H be the standard fBm. Then the limit

$$p(x) \equiv \lim_{T\to\infty} \frac{1}{T} \log\ P(\sup_{0\le t\le T} B_t^H \le x) \tag{7. 5}$$

exists for every x. Furthermore the function $p(.)$ is left-continuous and

$$p(x) = \sup_{T>0} \frac{1}{T} \log\ P(\sup_{0\le t\le T} B_t^H \le x). \tag{7. 6}$$

Since $E(B_0^H B_t^H) \ge 0$ for $t \ge 0$, Slepian's lemma is applicable and it implies that

$$P(\sup_{0\le t\le T_1+T_2} B_t^H \le x) \ge P(\sup_{0\le t\le T_1} B_t^H \le x)\ P(\sup_{T_1\le t\le T_1+T_2} B_t^H \le x)$$

for all $T_1, T_2 \ge 0$. From the stationarity of the process, it follows that

$$P(\sup_{T_1\le t\le T_1+T_2} B_t^H \le x) = P(\sup_{0\le t\le T_2} B_t^H \le x).$$

Hence

$$P(\sup_{0\le t\le T_1+T_2} B_t^H \le x) \ge P(\sup_{0\le t\le T_1} B_t^H \le x)\ P(\sup_{0\le t\le T_2} B_t^H \le x)$$

or

$$\log\ P(\sup_{0\le t\le T_1+T_2} B_t^H \le x) \ge \log\ P(\sup_{0\le t\le T_1} B_t^H \le x) + \log\ P(\sup_{0\le t\le T_2} B_t^H \le x) \tag{7. 7}$$

for all $T_1, T_2 \ge 0$. Existence of the limit function $p(.)$ follows from (7.7). Left-continuity of the function $p(x.)$ follows from the fact that $p(.)$ is non-decreasing and that the function $P(\sup_{0\le t\le T} B_t^H \le x)$ is continuous in x for every t. The representation given by (7.6) is again a consequence of the inequality proved in (7.7).$\diamondsuit$

Baumgarten (2011) studied the asymptotic behaviour of the probability that a stochastic process $\{Z_t, t \ge 0\}$ does not exceed a constant barrier up to time T when Z is the composition of two independent processes $\{X_t, t \in I\}$ and $\{Y_t, t \ge 0\}$. Baumgarten (2011) proved the following theorem.

Theorem 7.4: Let the process $\{B_t^H, -\infty < t < \infty\}$ be the centered fBm with Hurst index $H \in (0,1)$. Let $\{Y_t, t \ge 0\}$ be a self-similar process with Hurst index $\lambda > 0$ with continuous paths. Suppose that, for any $0 < \eta < 1$,

$$E[(\sup_{0\le t\le 1} Y_t)^{-\eta}] < \infty$$

and

$$E[(-\inf_{0\le t\le 1} Y_t)^{-\eta}] < \infty.$$

Then

$$P(\sup_{0\le t\le T} X(Y_t) \le 1) = T^{-\lambda+o(1)} \tag{7. 8}$$

as $T \to \infty$.

Remarks: Note that the rate of convergence in the equation (7.8) depends only on λ but not on H.

Consider the standard fractional Brownian motion $\{B_t^H, t \ge 0\}$ with Hurst index $H \in [0,1]$. Define the *storage process*

$$Y(t) = \sup_{\sigma\ge t}(B_\sigma^H - B_t^H - c(\sigma - t))$$

where $c > 0$ is a given constant. The process $\{Y(t), t \ge 0\}$ is stationary and

$$P(Y(0) > u) = P(\sup_{t\ge 0}(B_t^H - ct) > u).$$

Asymptotic behaviour of the tail distribution of the process Y was studied in Duffield and O'Connel (1996), Norros (1997), Narayan (1999) and Husler and Piterbarg (1999). Narayan used Fourier representations of the Brownian motion and fBm and and a similarity of their geometric properties. Husler and Piterbarg (1999) used another method called Double sum method (cf. Piterbarg (1999)). For detailed results, see Piterbarg (2001).

For any two given constants $c > 0$ and $\gamma \in [0,1]$, define a new process $\{J_\gamma(t), t \ge 0\}$ by

$$J_\gamma(t) = B_t^H - ct - \gamma \inf_{s\in[0,t]}(B_s^H - cs), t \ge 0. \tag{7. 9}$$

The process $\{J_\gamma(t), t \ge 0\}$ is called as a γ-reflected process with the fBm as input since it reflects at rate γ when reaching its minimum. If $\gamma = 1$, then the process J_1 is called the *workload process* or *queue length process* (cf. Awad and Glynn (2009)). In risk theory, J_γ can be interpreted as a *claim surplus process* since the surplus process of an insurance portfolio can be defined as

$$U_\gamma(t) = u + ct - B_t^H - \gamma \sup_{s\in[0,t]} (cs - B_s^H) = u - J_\gamma(t), t \geq 0$$

for any initial reserve $u \geq 0$. The process $\{U_\gamma(t), t \geq 0\}$ is called the risk process with tax payments of a *loss-carry-forward type* (cf. Asmussen and Albercher(2010)). Hashorva et al. (2013) studied the tail asymptotic behaviour of the process $M_\gamma(t), t \geq 0\}$, where

$$M_\gamma(T) = \sup_{t\in[0,T]} J_\gamma(t)$$

for $T \in (0, \infty]$. Husler and Piterbarg (1999) considered the case $T = \infty$. Debicki and Rolski (2002) and Debilcki and Sikora (2011) studied the results when $T \in (0, \infty)$. It is known that $M_1(t) \to \infty$ as $t \to \infty$ almost surely (cf. Duncan and Jin (2008)). For a complete discussion on these results, see Hashorva et al. (2013).

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